**On the classification of
arithmetic reflection groups
on hyperbolic 3-space**

Rudolf Scharlau

Bielefeld 1989

1. Introduction, survey of known results

The object of this paper are crystallographic reflection groups W on hyperbolic n-space H^n. Crystallographic means that W is a discrete group of isometries having a fundamental domain of finite volume. Reflection group means that W is generated by reflections in hyperplanes. If we choose the Klein model for H^n, that is, identify H^n with one of the two connected components of the set of vectors in $\mathbf{R}^{n+1}$ of norm -1 with respect to the quadratic form $-x_0^2 + x_1^2 + \dots x_n^2$,

$$H^n = \{(x_0, \dots, x_n) \in \mathbf{R}^{n+1} \mid x_0 \geq 1,\ -x_0^2 + x_1^2 + \cdots + x_n^2 = -1\},$$

the isometry group is the subgroup $O^+_{n,1}(\mathbf{R})$ of the orthogonal group $O_{n,1}(\mathbf{R})$ of the above form stabilizing H^n (i.e., preserving the condition $x_0 \geq 1$). A discrete subgroup $\Gamma \subset O^+_{n,1}(\mathbf{R})$ has a fundamental domain of finite volume on H^n if and only if the quotient space $O^+_{n,1}(\mathbf{R})/W$ has finite volume (with respect to the essentially unique $O^+_{n,1}(\mathbf{R})$-invariant measure). In particular this holds for all arithmetic subgroups $\Gamma \subset O^+_{n,1}(\mathbf{R})$ (see below).

The reflections in hyperplanes in this model are just ordinary linear reflections in hyperplanes $v^\perp \subseteq \mathbf{R}^{n+1}$ such that the vector v has positive norm, or equivalently, its orthogonal complement $v^\perp$ is again indefinite, necessarily of signature $(n-1,1)$, as a quadratic space. In the sequel, "reflection" will always mean a reflection of this particular kind. The reflection s_v belonging to v is given by the same well known formula as in the Euclidean case:

$$s_v(x) = x - \frac{2(v|x)}{(v|v)} v,$$

where $(x|y)$ is the scalar product $-x_0y_0 + x_1y_1 + \cdots + x_ny_n$.

The study of crystallographic hyperbolic reflection groups has been initiated in the 60s mainly by V. S. Makarov [Ma1, Ma2] and by E. B. Vinberg. (In the sequel, we shall omit the specification "crystallographic".) Vinberg's early papers [Vi1, Vi2] contain basic facts about the description of such groups by Gram matrices and "Coxeter-Vinberg"-diagrams and give a lot of examples.

A major problem which is not completely solved so far is the question of a full classification of hyperbolic reflection groups. In the present paper, we shall deal with the crucial case of the classification in dimension $n = 3$. (The case $n = 2$ is of a different nature, and the cases $n \geq 4$ will presumably rely on the case $n = 3$.)

In small dimensions, the classification problem is much more difficult than its "classical" counterpart of reflection groups on the sphere or on Euclidean space, a full treatment of which has been given by Coxeter already more than 50 years ago. The classification of finite (= spherical) and Euclidean reflection groups is in terms of the well known spherical and Euclidean (affine) Coxeter diagrams $A_n, B_n = C_n, D_n, \ldots, \tilde{A}_n, \tilde{B}_n, \tilde{C}_n, \tilde{D}_n, \ldots$ which play an important role in the theory of semisimple Lie groups and algebraic groups as well as in other areas. The main qualitative behaviour of this classification of spherical and Euclidean reflection groups can be summarized as follows:

– There are infinite series of such groups (contained in each other, with increasing dimension) but only few groups of a fixed dimension.

– The combinatorial structure of a fundamental domain is very simple: for irreducible groups it is necessarily a simplex, in general a product of simplices.

In the hyperbolic case, the behavior of reflection groups is essentially different. A basic theorem of Vinberg [Vi6] and Prokhorov [Pr], based on a method by Nikulin [Ni2], says that for large dimension there exist no such groups at all. In particular, there are no infinite series. On the other hand, for small n there exist lots of groups, and the fundamental domain which combinatorially still is an ordinary convex polytope, can be very complicated. By this we mean in particular that the number of faces, i.e. the number of generators of the group can be very large compared to the dimension. For $n = 3$, it is not even known whether the number of (conjugacy classes of) hyperbolic reflection groups is finite.

An interesting subclass where certain finiteness results do hold are the arithmetic hyperbolic reflection groups. A discrete subgroup $\Gamma \subset O_{n,1}(\mathbf{R})$ is called arithmetic if it is commensurable to some integral orthogonal group $O(f, \mathbf{o}_K)$, where $\mathbf{o}_K$ is the ring of integers of an algebraic number field and f is a quadratic form over $\mathbf{o}_K$. (Commensurable means that the intersection has finite index in both groups.) The condition on K and f for $O(f, \mathbf{o}_K)$ to be a discrete subgroup of $O_{n,1}(\mathbf{R})$ is that $K \subseteq \mathbf{R}$ be totally real, f of signature $(n, 1)$, and σf (positive or negative) definite for all embeddings $\sigma \neq id$ of K into $\mathbf{R}$. The arithmetic reflection groups on hyperbolic space were studied for the first time in [Vi1].

A useful technical remark is that if an arithmetic group $\Gamma = W$ is generated by reflections (in the above sense), then it is actually contained in (not only commensurable to) an orthogonal group $O(f_W, K_W)$, where the form f_W and the field K_W are derived from the Gram matrix or Coxeter-Vinberg-diagram of W [Vi1, Ni2]. The form $f = f_W$ has the property that the canonical normal subgroup $W(f) \subseteq O(f)$ generated by all reflections s_v preserving f (i.e. leaving invariant the lattice $\subseteq \mathbf{R}^{n+1}$ defining f) has finite index. Indeed, the given group W is necessarily contained in $W(f)$. Conversely, if f is such that the above signature condition holds and $W(f)$ is of finite index in $O(f)$, then $W(f)$ is an arithmetic hyperbolic reflection group. Here we have to recall that an arithmetic group always has a fundamental domain of finite volume.

We see that a slightly weaker version of the classification problem for arithmetic hyperbolic reflection groups can be stated as follows:

– For which integral quadratic forms f over a totally real number field K subject to the above signature conditions does the reflection subgroup $W(f) \subseteq O(f)$ have finite index?

Following Vinberg [Vi4], we shall call such a form reflective or a reflection form.

Stated in this way, the problem is certainly interesting in its own right, and thus independent from the theory of hyperbolic reflection groups. Indeed, there exist a great deal of research on the question of generation by reflections of orthogonal groups over rings, starting with the classical theorem of Witt that an orthogonal group over a field is always generated by reflections. For large classes of local and more generally semilocal rings, including rings of **p**-adic integers (complete discrete valuation rings) the answer is "in general" positive. Using Kneser's Strong Approximation Theorem, these results can be globalized to quadratic forms over number fields of real rank ≥ 2, i.e. the sum of the Witt indices for the real places, is at least 2. See Kneser [Kn4] for a conclusive treatment which brings down the number of exceptions to a minimum.

These results do not carry over to orthogonal groups of real rank 1. Indeed, a basic theorem by Nikulin [Ni3, Theorem 5.2.1] states that for fixed degree of the ground field K, the number of (similarity classes of) of reflection forms f is finite. If $n \geq 10$, the restriction on the ground field is not needed [Ni3, § 4, and Appendix]. For $n < 10$, it is an open question whether or not there are infinitely many ground fields. Nikulin's proof is geometrical; starting from known combinatorial properties of the fundamental polyhedron it gives bounds on the entries of the Gram matrix and the conjugates of generating elements of K. The arithmetricity only enters at the end of the proof. It is important to recognize that Nikulin's work does not give a method of determining the finitely many forms in question. The list of candidates that comes from Nikulin's proof is very large, and for a given form f there seems to be no a priori criterion to decide whether $[O(f) : W(f)]$ is finite or infinite.

Here, we should mention that Nikulin's interest in the problem originally came from complex algebraic geometry, where in connection with K3-surfaces the problem of determining all forms f such that $O(f)$ is generated by reflectiond s_v such that $(v|v) = 2$ occurs. This problem had been solved (for dimension $n + 1 \geq 5$) by Nikulin in another paper [Ni1]; the proof uses the connection with algebraic surfaces and highly nontrivial facts about these. Our present problem is more general, and from the point of view of hyperbolic geometry or of the theory of integral quadratic forms, there is no reason to restrict to reflections on vectors of norm 2.

The principal message of this paper is that the theory of integral quadratic forms gives the tools for the classification of arithmetic hyperbolic reflection groups, rather than the general theory of hyperbolic reflection groups, or their relationship to other areas.

So far, a full classification of reflective forms has only been given for unimodular forms over $\mathbf{Z}$, $\mathbf{Z}[(1+\sqrt{5})/2]$, and $\mathbf{Z}[\sqrt{2}]$ [Vi4, Mey, Vi-Ka, Bu1, Bu2]. That is, the maximal dimensions for which these forms are reflective have been determined. The results on "Bianchi groups" in [Šv] are of a similar nature, however they only apply to a rather restricted class of groups which in particular are not maximal.

The only other results on the classification of hyperbolic reflection groups are under very restrictive assumptions on the combinatorial type of the fundamental domain: if this is a simplex or a simplicial prism, the classification in terms of the Gram matrices is known [La, Ko, Ka]. Using Vinberg's criteria [Vi1] one can then a posteriori easily determine which of these groups W are arithmetic, and one can also embed them canonically into orthogonal groups $O(f_W, \mathbf{o}_{K_W})$ (see above). This at least gives some more examples (in addition to the unimodular cases) also in small dimensions, in particular for $n = 3$. Other examples are contained in [Vi1, Vi2, Men1, Ša] and in [E-G-M], where forms $f_d = x_0x_1 + x_2^2 + dx_3^2$ are studied. In addition to those forms given in [E-G-M], Mennicke has some unpublished examples of values d such that f_d is reflective and the fundamental domain of f_d is combinatorially complicated.

The contents of this paper is as follows.
We have two main assumptions:

- The dimension of our hyperbolic space is equal to three: $n = 3$.
- We are only considering groups W such that H^n/W is not compact.

The second assumption means that we need only to consider forms f such that

- f is defined over $\mathbf{Z}$ and isotropic over $\mathbf{Z}$.

Indeed, by a well known and fundamental result of Borel and Harish-Chandra [Bo-Ha] or Mostow and Tamagawa [M-T], $O(f, \mathbf{o}_K)$ is cocompact in $O(f, \mathbf{R})$ if and only if f is isotropic, and if $K \neq \mathbf{Q}$, then there are real embeddings of K for which f is definite, so f is necessarily anisotropic.

Most reflection forms known by now are isotropic, therefore it is natural to begin a full classification with this case. Moreover, in the isotropic case, the combinatorial structure can be more complicated and more interesting than in the anisotropic case. This comes from the presence of "infinite vertices" or "cusps" in the non-compact case. Whereas in an ordinary vertex always 3 edges meet, in a cusp 3 or 4 edges may meet, the second case belonging to a stabilizer with diagram

$$\circ \overset{\infty}{—} \circ \quad \circ \overset{\infty}{—} \circ .$$

On the other hand, anisotropic reflective quadratic forms do exist as well, and their classification is equally desirable. For small ground fields, i.e. $K = \mathbf{Q}$, or K real quadratic of small discriminant, a classification seems to be within reach along the general lines of the present paper. We shall come back to this question in future work. The general problem with unrestricted ground field at present seems to be intractable, as was remarked above.

Because the list of isotropic reflection forms is very large, it seems reasonable to introduce one further natural condition:

– The group $W(f)$ is maximal among groups of this kind.

Inside the class of reflection group $W(f)$, the restriction to maximal groups is only technical. The classification of arbitrary isotropic reflection forms appears to be possible by the general methods of this paper. For the class of all arithmetic reflection groups, the maximality assumption is essential. We do not know how to classify all reflection subgroups of a given maximal group. We do not even know whether or not one of the maximal groups determined in this paper contains reflection subgroups of arbitrarily large finite index.

Concerning the maximality condition, we also want to make the following remark: If one has some classification of reflection groups in terms of Gram matrices or Coxeter-Vinberg-Diagrams (for instance the above mentioned groups with fundamental domain a simplex or a prisma), it is usually not at all clear which groups embed into which, in particular, which are maximal. Therefore, even if one is finally interested in the full classification of reflection forms and not only into those ones with maximal groups, it is in any case desirable to know which groups are maximal, and more generally to have arithmetic criteria which groups can be embedded into each other.

We now can formulate the main result of this paper: In § 3, we give a full set of representatives for the non-cocompact maximal reflection groups on H^3, or equivalently, for the similarity classes of isotropic reflective quadratic forms f over $\mathbf{Z}$ such that $W(f)$ is maximal. It turns out that there are precisely 49 such groups, and for each of these, there is a more or less canonical quadratic form f.

The main combinatorial properties of the 49 fundamental domains are collected in the Table at the end of § 2. The maximum number of faces is 20, the maximum number of vertices (including cusps) is 34, and the maximum number of edges of one face is 12. In an Appendix we give, for each of the 49 forms f, the list of fundamental roots and thus the full information about the group $W(f)$. For the reader's convenience, we also tabulate some derived data (Gram matrices, vertices, cusps).

In § 2, we collect various preparatory propositions. We derive consequences from the maximality assumption, and we work out some general observations on reflective forms due to E. B. Vinberg and J. Mennicke. This Section ends with a finite list of 324 forms which contains representatives for all reflective forms. Using Vinberg's well known algorithm [Vi4, cf. N-S, C-S, chap. 28] it is now easy to show that 51 of these forms are reflective, and to derive the Coxeter-Vinberg diagram of $W(f)$. Of course, this is done with the aid of a computer. It turns out that the 51 forms give rise to 49 different groups. The main problem now is to prove that f is not reflective in the remaining cases. This is achieved by two Propositions 3.1 and 3.2, together with Vinberg's algorithm. Our criterion is explicitely stated in Proposition 3.3. Although in the end only well known methods from the arithmetic theory of quadratic forms and from the theory of Coxeter groups are used, this criterion seems to be new. The idea is to distinguish $W(f)$ from $O^+(f)$ by comparing the actions on the reflections s_v, for certain specified values $f(v)$. After Proposition 3.3 and the preparatory results of § 2, the proof of the Main Theorem is straightforward — modulo extensive use of a computer.

In this paper, a good knowledge of integral quadratic forms and of reflection groups is presupposed. Results which are contained for instance in [OM, Kn2] or in [Bou, Vi1], respectively, are usually used without explicit reference.

Acknowledgements. I am indebted to F. Grunewald, U. Rehmann, and R. Schulze-Pillot for helpful discussions. My thanks also go to Sylvia Himpel who wrote a computer program which checks the condition stated in Proposition 2.2.

2. Necessary conditions for reflective forms

In this Section, we collect various necessary conditions on a quadratic lattice L of signature $(3,1)$, isotropic over $\mathbb{Q}$, in order that the reflection subgroup $W(L) \subseteq O^+(L)$ be a maximal arithmetic reflection group on hyperbolic 3-space H^3.

We start by deriving consequences from the maximality assumption. The resulting Proposition deals with the full orthogonal group $O(L)$, the reflections do not play a rôle at the moment.

Proposition 2.1. If $O(L)$ is maximal, then we may assume that L has the shape $L = \mathsf{H} \perp {}^s[a,b,c]$, where H denotes the hyperbolic plane $(\mathbb{Z}^2, xy)$, ${}^s[a,b,c]$ the binary lattice $(\mathbb{Z}^2,\ s(ax^2+bxy+cy^2))$, the discriminant $-D_0 = -(4ac-b^2)$ of $[a,b,c]$ is a fundamental discriminant (discriminant of a quadratic field), and s is a squarefree natural number relatively prime to D_0.

Proof. By iterated application of the substitution $L_{(p)} \mapsto p^{-1}(L_{(p)} \cap p^2 L^{\#}_{(p)}) + L_{(p)}$, $L_{(p)} = \mathbb{Z}_p L$, for the prime divisors p of $detL$, we may assume that locally L is "p-elementary", i.e. $pL^{\#}_{(p)} \subseteq L_{(p)}$, or equivalently, L has only unimodular and p-modular Jordan components, for all p. See e.g. [Wa1; Wa2; Vi6, § 4] for details. We shall now deal with a fixed prime number and shall write again L instead of $L_{(p)}$. If $L = L_0 \perp {}^p L_1$ is a Jordan decomposition, then ${}^p L^{\#} = {}^p(L_0 \perp p^{-1\,p} L_1) = {}^p L_0 \perp p^{-1\,p^2} L_1 \cong {}^p L_0 \perp L_1$. That is, we may interchange L_0 and L_1, in particular we may assume $dimL_1 \leq dimL_0$. We shall now show that we can achieve that the hyperbolic plane H embeds into L. Once this is proved locally, it holds globally, since L has only one class in its genus. First consider the case $p \neq 2$. If $dimL_0 \geq 3$, it is clear that H embeds into L_0. If $dimL_0 = dimL_1 = 2$, then one of L_0, L_1 is isotropic, since $L_0 \perp {}^p L_1 = L$ is isotropic. After possibly interchanging L_0 and L_1, we may assume that $L_0 \cong \mathsf{H}$.

Now we consider the case $p = 2$. There are various cases according to L_0, L_1 are even or odd. In some cases, we have to pass from L to the even sublattice $\mathring{L} = \{x \in L \mid f(x) \in \mathbb{Z}_2\}$. This may introduce a onedimensional 4-modular Jordan component, but this has been admitted in the statement of the Proposition, since $D_0 \equiv 0(8)$ is possible. We only give the proof in the most difficult cases $dimL_0 = dimL_1 = 2$. If L_0, L_1 are both even, then the same proof as for $p \neq 2$ applies. If one of them is even, the other is odd, then we may assume that L_0 is even and L_2 is odd, that is, L_2 is a diagonal form $\langle a,b \rangle$, a,b odd. If $L_0 \not\cong \mathsf{H}$, then $L_0 \cong A_2$, the A_2-root-lattice with matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. We now show, using the general rules on the non-uniqueness of 2-adic Jordan decompositions that, given a,b odd, there exist a', b' such that

$$A_2 \perp {}^2\langle a,b \rangle \cong \mathsf{H} \perp {}^2\langle a',b' \rangle.$$

In the notation of [C-S], Chap.15, this reads

$$1_{II}^{-2} 2_t^{\varepsilon,2} \cong 1_{II}^{+2} 2_{t'}^{\varepsilon',2},$$

where $t = a + b$, $t' = a' + b'$, $\varepsilon = \left(\frac{ab}{2}\right)$, $\varepsilon' = \left(\frac{a'b'}{2}\right)$. It is proved for instance by using Theorem 10 of loc. cit. that this holds for $a' := a$, $b' := b + 4 \pmod 8$. If finally L_0, L_1 are both odd, then we may pass to the even sublattice $\overset{\circ}{L}$, which is of the shape ${}^2M$, for M odd, unimodular.

Once it is proved that L splits off H, the precise shape as given in the proposition comes from the particular Jordan structure. □

In view of our final goal, it has to be shown that the groups given in Proposition 2.1 are actually maximal, and one has to know when two such groups are conjugate. We shall not give a proof here, but restrict ourselves to some remarks. First of all, is a fact that the lattices L as described in the Proposition have a group which is locally everywhere maximal, that is, $O(L_{(p)})$ is maximal for all p. Indeed, for $p \neq 2$ it is probably well known that a local orthogonal group is maximal if and only if the lattice is p-elementary. For $p = 2$, this is not quite true since the even sublattice $\overset{\circ}{M}$ may have a larger group but still be 2-elementary. We do not know of any "handy" reference which states what happens precisely, but we hope to come back to this point in future work. An answer to the local maximality question in a general setting is given by the theory of Bruhat-Tits buildings where the maximal compact subgroups of any reductive group are described in terms of an affine root system. In particular, it must be possible to derive the result stated above from the paper [B-T]; cf. also [Ti3]. Concerning the question of global maximality, that is, maximality of the arithmetic group $O(L)$ itself, the reader may consult [Bo, Section 4] for the case of groups acting on H^3, and [Ro1, Ro2] for some general theory.

Concerning the question of conjugacy for the groups described in Proposition 2.1, there is one situation where it actually happens that two nonisometric forms give rise to conjugate groups. Consider lattices $\mathsf{H}\perp{}^s[a,b,c]$, $\mathsf{H}\perp{}^{s'}[a',b',c']$ as above, and assume that $s = s'$ is such that $D_0 = 4ac - b^2 = 4a'c' - b'^2$, and $-D_0$ is a square mod p, resp. mod 8 if $p = 2$, for all prime divisors $p|s$. Assume furthermore that ${}^s[a,b,c]$ is isometric to $[a',b',c']$ over the rationals, that is, ${}^s[a,b,c]$ represents rationally at least one number represented by $[a',b',c']$. We claim that $\mathsf{H}\perp{}^s[a,b,c] \cong {}^s\mathsf{H}\perp[a',b',c']$. It is sufficient to see this locally everywhere. For $p \nmid s$, we obviously have $\mathsf{H} \underset{\mathbb{Z}_p}{\cong} {}^s\mathsf{H}$, and ${}^s[a,b,c] \underset{\mathbb{Z}_p}{\cong} [a',b',c']$ by the above assumption. If $p|s$, then $\mathsf{H}\perp{}^s[a,b,c] \underset{\mathbb{Z}_p}{\cong} \mathsf{H}\perp{}^s\mathsf{H} \underset{\mathbb{Z}_p}{\cong} [a',b',c']\perp\mathsf{H}$, by the assumption on D_0 mod p. If we dualize the lattice $L = {}^s\mathsf{H}\perp[a',b',c']$ with respect to all $p|s$, that is, define a new lattice N by $N\mathbb{Z}_p = (L\mathbb{Z}_p)^\#$ if $p|s$, $N\mathbb{Z}_p = L\mathbb{Z}_p$ if $p \nmid s$, we have $N \cong {}^{s^{-1}}\mathsf{H}\perp[a',b',c']$, since $[a',b',c']$ is unimodular with respect to the primes p dividing s. Thus $O(\mathsf{H}\perp{}^s[a,b,c]) \cong O(L) = O(N) = O({}^sN) \cong O(\mathsf{H}\perp{}^s[a',b',c'])$, as claimed.

From now on, we may assume that our lattices L are of the type described in Proposition 2.1, and we come back to the question of generation by reflections. We collect necessary conditions on L for $W(L)$ to be of finite index in $O(L)$.

The first and in a sense most essential condition on reflective quadratic forms is the following basic result of Vinberg's concerning positive definite orthogonal direct summands of reflective forms. We recall that a **root** of a quadratic lattice L is a primitive vector v such that the reflection s_v maps L into itself. A definite lattice is called **reflective** if the sublattice generated by the roots (the root lattice of L) has finite index in L.

Lemma 1 (Vinberg [Vi3]). In order that a lattice $\mathsf{H}\perp M$, H the hyperbolic plane, M a positive definite lattice, be reflective, a necessary condition is that M is reflective. □

Since a lattice of the shape $\mathsf{H}\perp M$ has only one class in its genus, Vinberg's lemma can be sharpened to the statement that every form in the genus $genM$ is reflective if $\mathsf{H}\perp M$ is reflective.

We shall now work out this criterion in the case of a binary $M = [a,b,c]$ as above. Let $[a,b,c]$ be reduced, that is $|b| \le a \le c$ and $b > 0$ if $a = |b|$ or $a = c$. We claim that $[a,b,c]$ is reflective if and only if it is of one of the following shapes:

$$[a,0,c]$$
$$[a,a,c]$$
$$[a,b,a]$$

Indeed, if this is not the case, then the minimum a of $[a,b,c]$ is obtained only at $\pm(1,0)$, the second minimum c is only attained at $\pm(0,1)$, and now it follows from $b \ne 0$ that $-id$ is the only nontrivial isometry. If, conversely, $[a,b,c]$ is as above, then the set of roots of $[a,b,c]$ is easily seen to be the following:

$[a,0,c],\ a \ne c$: $\pm(1,0)_a, (\pm 0,1)_c$

$[a,a,c],\ a \ne c$: $\pm(1,0)_a, \pm(-1,2)_{4c-a}$

$[a,b,a],\ a \ne b$: $\pm(1,1)_{2a+b}, \pm(-1,1)_{2a-b}$

$[a,0,a]$: $\pm(1,0)_a, \pm(0,1)_a, \pm(1,1)_{2a}, \pm(-1,1)_{2a}$

$[a,a,a]$: $\pm(1,0)_a, \pm(0,1)_a, \pm(-1,2)_{3a}, \pm(2,-1)_{3a}, \pm(1,1)_{3a}, \pm(-1,1)_a$.

(The subscript indicates the value of the quadratic form.)

In view of the above results on the maximality of orthogonal groups $O(\mathsf{H}\perp[a,b,c])$, we shall now assume that the binary form has the shape ${}^{s}[a,b,c]$, where $-D_0 = b^2 - 4ac$ is a field discriminant, s is squarefree and $gcd(s, D_0) = 1$.

The following Proposition had been obtained for the special of "Bianchi groups" (that is, the binary form $[a,b,c]$ is in the principal genus) independently (and prior to our result) by E. B. Vinberg [Vi7].

Proposition 2.2. Let $-D_0$ be the discriminant of an imaginary quadratic field, denote by $\mathcal{C}_{D_0}$ the class group of $\mathbf{Q}(\sqrt{-D_0})$. If $\mathsf{H}\perp^{s}[a,b,c]$ is reflective, for some binary form $[a,b,c]$ of discriminant $-D_0$, and some s relatively prime to D, then $\mathcal{C}^4_{D_0} = \{1\}$.

Before proving this proposition, we have to recall a few very classical facts about the relation between binary quadratic forms and ideal classes in quadratic fields, about genera of binary forms and about ambiguous forms. See e. g. [B-S, Za].

Let D_0 be as above, $K = \mathbf{Q}(\sqrt{-D_0})$. If $\mathbf{a} \subseteq K$ is a fractional ideal, then the mapping

$$\mathbf{a} \longrightarrow \mathbf{Z}$$
$$\alpha \mapsto N\alpha / N\mathbf{a} =: F_{\mathbf{a}}(\alpha)$$

where N denotes the norm, is a (primitive) quadratic form of discriminant $-D$. Two ideals $\mathbf{a}$ and $\mathbf{b}$ give rise to equivalent forms if and only if they lie in the same ideal class. Here, equivalence of forms (M, f) and (M', f') means "proper equivalence", that is, the modules M, M' are oriented and an isometry is supposed to be orientation preserving, i. e. to have determinant $+1$ with respect to positively oriented bases. A basis (α, β) of an ideal is positively oriented if $Re(\beta/\alpha) > 0$. Every form of discriminant $-D_0$ is isometric to such a norm form of an ideal. An ideal class $[\mathbf{a}]$ resp. the corresponding quadratic form is called **ambiguous** if it is invariant under the nontrivial automorphism of K, or equivalently, $[\mathbf{a}]^2 = [1]$ in the class group $\mathcal{C}$. This holds if and only if the corresponding form $[a,b,c]$ is properly equivalent to $[a,-b,c]$. Since $[a,b,c]$ is assumed to be $|b| \le a \le c$, and $b > 0$ if $a = |b|$ or $a = c$, this can only hold if $b = 0$ or $a = |b|$ or $a = c$. Thus we have proved the following Lemma.

Lemma 2. A binary form is reflective if and only if it is ambiguous. $\square$

Proof of Proposition 2.2: We may restrict ourselves to the case $s = 1$. If $F_{\mathbf{a}}$, $[\mathbf{a}]^2 = [1]$ is an ambiguous form such that the whole genus $F_{\mathbf{b}}$, $[\mathbf{b}] \in [\mathbf{a}]^2\mathcal{C}$ of $F_{\mathbf{a}}$ consists of ambiguous forms, then $[\mathbf{b}]^2 = [1]$ for all such $\mathbf{b}$, and necessarily $\mathcal{C}^4 = \{[1]\}$. $\square$

The next Lemma deals with the stabilizer $O^+(L, v)$ in $O^+(L)$ of a root v. It acts on the orthogonal complement $\mathbf{R}v^{\perp}$ which can be thought of as a hyperbolic plane. Here, it is only supposed that L is of signature $(3,1)$.

Lemma 3 (Mennicke [E-G-M, § 11]). If L is reflective and v is a root of L, then the Fuchsian group $SO^+(L, v)$ is of genus zero. □

For the class of lattices L considered in this Section, some of the groups $SO^+(L, v)$ occuring in Lemma 3 are now exhibited explicitly.

Lemma 4. If $L = \mathsf{H}\perp {}^s[a, b, c]$ is as above, $D = s^2(4ac - b^2)$, and furthermore $[a, b, c]$ is reflective, then there exists a root v of L and a squarefree integer m such that

$$SO^+(L, v) \cong SO^+(\mathsf{H}\perp\langle 2m\rangle),\ \ m \geq \sqrt{D}/2.$$

Proof: First of all, we claim that $SO^+(v, L)$ can be identified with the group in three variables $SO^+(v^\perp \cap L)$. That is, if $\phi \in SO^+(v^\perp \cap L)$ is given, the extension to all of $L \otimes \mathbf{R}$ defined by $\phi v = v$ maps L into itself. From the fact that v is a root, it readily follows that the sublattice $\mathbf{Z}v\perp(v^\perp \cap L)$ is of index 1 or 2 in L. If the index is 1, our claim is obvious. If the index is 2, the assertion is readily checked by ad-hoc-arguments, using the particular structure of L.

Consider roots of the form $(0, 0, x, y)$, where (x, y) is a root of the binary lattice $[a, b, c]$, and let us call these roots "standard roots" for short. Recall the above list of the roots (x, y) for the various kinds $[a, 0, c]$ etc. of binary reflective forms. This list immediately shows that we may take for v a short standard root. Only in the case $[a, 0, a]$ we agree to take for v a long standard root, $f(v) = 2a$. Then $v^\perp \cong \mathsf{H}\perp\mathbf{Z}v'$, where in all cases v' is a long standard root. Thus, $SO^+(v, L)$ is a desired, and $m = f(v')$. For the forms ${}^s[a, 0, c]$ we have $D = 4s^2ac$, that is $m = sc \geq \sqrt{D}/2$, as claimed. In the other cases, the estimate is even better. □

The next Lemma, together with the two previous Lemmata, gives the explicity finiteness result which is the basis of the precise determination of all reflective L, subject to our standard conditions.

Lemma 5. The Fuchsian group $SO^+(\mathsf{H}\perp\langle 2m\rangle)$, m squarefree, has genus zero precisely for the values $m = 1, 2, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 33, 34, 35, 38, 39, 41, 42, 46, 47, 51, 55, 59, 62, 66, 69, 70, 71, 78, 87, 94, 95, 105, 110, 119$.

Sketch of Proof: Helling in [He] gives a formula $g(m)$ for the genus in question which fist of all allows to check the result within any reasonable finite range like $m \leq 1000$. By the shape of this formula, one has an estimate of the form $g(m) \geq C_1 \cdot m - C_2 \cdot h(m)$, where $h(m)$ is the class number of $\mathbf{Q}(\sqrt{-m})$, for certain explicit positive constants C_1, C_2. Substituting an estimate $h(m) \leq C_\varepsilon m^\varepsilon$ for some ε strictly between $\frac{1}{2}$ and 1, and an explicit C_ε, one arrives at moderate bounds on the possible m with $g(m) = 0$. The possibility of giving an explicit bound on the values m had already been stated in [E-G-M] (for prime numbers m). I am indebted to the second author of [E-G-M] for clarifying this point to me. □

If we combine all results of this Section, we see that any maximal non-cocompact arithmetic reflection group on H^3 is among the groups $W(L)$, where L is as follows:

$$L = \mathbf{H}\perp {}^s[a,b,c]$$
$$|b| \le a \le c$$
$$b = 0 \text{ or } a = c \text{ or } b = c$$
$$-D_0 = b^2 - 4ac \text{ a fundamental discriminant}$$
$$D = s^2(4ac - b^2) \le 56644,$$
$$\mathcal{C}^4 = 1, \text{where } \mathcal{C} \text{ is the class group of } \mathbb{Q}(\sqrt{-D_0}),$$
$$f(v_1), \ldots, f(v_r) \text{ is one of the numbers in Lemma 5,}$$
$$\text{where } [a_i, b_i, c_i],\ i = 1, \ldots, r \text{ runs over the genus of}$$
$$[a,b,c] = [a_1, b_1, c_1], \text{ and } v_i \text{ is a long root of } {}^s[a_i, b_i, c_i].$$

It is explained for instance in [B-S, Chap.3, § 8, Exerc.25; Za, § 12, p. 118 f.] how to verify the condition $\mathcal{C}^4 = 1$. Using this, it is lengthy but straightforward to write a computer program which produces all $\mathbf{H}\perp^s[a,b,c]$ subject to the above conditions. It turns out that there are precisely 324 of them.

To each of these forms, Vinberg's algorithm for finding fundamental roots is applied. We briefly recall this procedure from [Vi4, § 3] and give the relevant formulae in our particular case.

- Fix a point $p_0 \in H^3$ having a stabilizer in $W(L)$ of rank 3. We take $p_0 = (-1,1,0,0)$.
- Fix a basis v_1, v_2, v_3 of the root system of the positive definite lattice $p_0^\perp \cap L$. (That is, the reflections $s_{v_1}, s_{v_2}, s_{v_3}$ generate the stabilizer of p_0 in $W(L)$). We take $v_1 = (-1,-1,0,0)$, and v_2, v_3 of the form $(0,0,x_2,y_2)$, $(0,0,x_3,y_3)$, where (x_2,y_2), (x_3,y_3) are fundamental roots of ${}^s[a,b,c]$ (see above).
- If $v_1, \ldots, v_k$ are already chosen, for some $k \ge 3$, then v_{k+1} is a root v subject to the following conditions:
  (1) $(v \mid p_0) \ge 0$.
  (2) $(v \mid v_i) \le 0$ for $i = 1, \ldots, k$.
  (3) $(v \mid p_0)^2 / f(v)$ is minimal among all roots v satisfying (1) and (2).

The function $(v \mid p_0)^2/f(v)$ minimizes the distance between the point p_0 and the plane $v^\perp$.

For a root v, we denote by H_v^+ the halfspace $\{p \in H^3 \mid (v \mid p) \geq 0\}$. If $v_1, \ldots, v_k$ are as above, we set $P_k := \bigcap_{i=1}^{k} H_{v_i}^+$. It is readily derived from the Gram matrix of the vectors $v_1, \ldots, v_k$ whether P_k has finite volume; the condition is that P_k, including its cusps (vertices at infinity) combinatorially is a compact convex polytype (cf. [Vin4] and the Appendix of this paper). If this is the case, for some k, then a root $v = v_{k+1}$ subject to (1), (2), (3) does not exist (the algorithm stops), and P_k is a fundamental domain for $W(L)$. This happens (for various values of k between 4 and 20) for the 49 forms listed in the Theorem of Section 3 below, and also for the forms $\mathsf{H}\perp^2[2,1,1]$ and $\mathsf{H}\perp^3[2,2,3]$ which, however, have the same orthogonal group as $\mathsf{H}\perp^2[1,1,4]$ and $\mathsf{H}\perp^3[1,0,5]$ respectively (see above). In the remaining cases, the number $k = k_C$ of roots $v_1, \ldots, v_k$ as above, and in addition satisfying the bound

$$(v_i \mid p)^2/f(v_i) \leq C$$

numerically increases with C. This suggests that k_C tends to infinity with C which is equivalent to $W(L)$ being of infinite index in $O^+(L)$. In the next Section, we give a criterion which allows to actually prove this.

3. The Main Theorem

We immediately proceed to the statement of our main result. We recall the notation H for the hyperbolic plane over $\mathbf{Z}$, and ${}^s[a,b,c]$ for the binary form $s(ax^2+bxy+cy^2)$ on $\mathbf{Z}^2$.

Theorem. The groups $W(L)$, for the following 49 quadratic lattices $L = \mathsf{H}\perp{}^s[a,b,c]$ are a complete set of representatives for the maximal arithmetic reflection groups with noncompact fundamental domain on hyperbolic 3-space:

$[1,1,1]$	$[1,0,1]$	$[1,1,2]$	$[1,0,2]$	$[1,1,3]$
${}^2[1,1,1]$	$[1,1,4]$	$[2,1,2]$	$[1,1,5]$	$[1,0,5]$
$[2,2,3]$	$[1,0,6]$	$[2,0,3]$	${}^2[1,1,2]$	$[3,1,3]$
${}^3[1,0,1]$	$[1,1,10]$	$[1,0,10]$	$[2,0,5]$	${}^2[1,1,3]$
$[1,0,13]$	$[1,0,14]$	${}^2[1,1,4]$	${}^3[1,1,2]$	$[1,0,17]$
${}^3[1,0,2]$	${}^5[1,1,1]$	$[1,0,21]$	$[3,0,7]$	$[5,4,5]$
${}^5[1,0,1]$	$[1,0,30]$	$[2,0,15]$	$[3,0,10]$	$[5,0,6]$
$[1,0,33]$	${}^2[3,1,3]$	${}^7[1,1,1]$	$[3,0,14]$	$[6,0,7]$
${}^3[1,0,5]$	${}^7[1,0,1]$	${}^5[1,0,2]$	${}^{10}[1,1,1]$	${}^3[1,0,10]$
${}^3[2,0,5]$	${}^{13}[1,1,1]$	${}^{14}[1,1,1]$	${}^{15}[1,0,1]$	

More detailed information about the 49 groups is contained in the Table given below, and in the Appendix.

In view of the results of § 2, we make the following assumptions for the rest of this section:

$L = (\mathbf{Z}^4, f)$ is a quadratic lattice of the shape $\mathsf{H}\perp{}^s[a,b,c]$ such that $-D_0 = -(b^2-4ac) < 0$ is a field discriminant, $gcd(D_0,s) = 1$, $[a,b,c]$ is reduced, and $b=0$ or $a=b$ or $a=c$.

The following Proposition deals with the orbits of the orthogonal group $O^+(L)$ on the roots of fixed length, or equivalently, with the conjugacy classes in $O^+(L)$ of reflections with a fixed spinor norm. In the context of this paper, this Proposition is just used as a Lemma which will allow us to "distinguish" $W(L)$ from $O^+(L)$. However, the determination of conjugacy classes of reflections is a general result on the group $O^+(L)$ which is of some obvious independent interest.

Under the general theme of "Witt's Theorem over rings", there is a lot of literature on the question when two vectors of the same length in a quadratic module are conjugate under the orthogonal group, or more general, when an isometry between two submodules can be extended. The papers [Hs1, Hs2, Ja, Kn3, So, Tr] are only an incomplete set of references. Of course, the proof of our Proposition 3.1 relies on such general results. The only "concrete" result known to us, giving an explicit determination of the conjugacy classes (under certain conditions), is Theorem 11.3 in [E-G-M]. Our result is a generalization of that Theorem. Mennicke's proof is unpublished, and different from our proof. Apparently, such "concrete" results can only be expected under special hypotheses, for instance that the lattices are p-elementary for all primes p, or that the vectors in question have particular properties like beeing roots.

Proposition 3.1. If D_0 is odd or if $8 \nmid D_0$ and $f(v)$ is even, then any two roots v, v' such that $f(v) = f(v') > 0$ are conjugate under $O^+(L)$. In any case, there are at most two conjugacy classes of roots v for fixed $f(v) > 0$.

Proof: One first proves the result locally, i.e. over the p-adic integers $\mathbf{Z}_p$, for all p. If $p \neq 2$, it is easily seen that a root v splits off, $L = \mathbf{Z}v \perp v^\perp$. The result follows from the well known uniqueness of orthogonal complements over $\mathbf{Z}_p$. Now consider our lattice over $\mathbf{Z}_2$. In addition to the value $q = f(v)$, also the number e_v defined by $(v|L) = 2^{e_v}\mathbf{Z}_2$ is an invariant of the conjugacy class. Since L has 2^i-modular Jordan components at most for $i = 0,1,2$, and roots v are primitive by definition, we have $e_v = 0,1,2$, and $e_v = 2$ can only occur if $D_0 \equiv 0(8)$. Recall furthermore that, since v is a root, $2^{e_v} \in \mathbf{Z}_2 q$, that is, the 2-exponent of q is at most e_v.

We now distinguish 3 cases.

Case 1. D_0 odd. Then L is even unimodular if s is odd, or L has a Jordan decomposition $L_0 \perp {}^2L_1$ with L_0, L_1 even binary unimodular. One then deduces for instance from Kneser's criterion in [Kn2, §2] that v and v' are conjugate if v, v' are roots such that $e_v = e_{v'}$. Thus, in order to prove that there is only one orbit, we have to show that e_v is determined by the value $q = f(v)$. If q is even, it is clear by the general observations above that $e_v = 1$. Conversely, if $e_v = 1$, then $v \in {}^2L_1$ for an appropriately chosen Jordan decomposition $L = L_0 \perp {}^2L_1$, and since L_1 is an even lattice for any such Jordan decomposition, necessarily $q = f(v)$ is even. Thus, if q is odd, then $e_v = 0$ is indeed determined by q.

Case 2. $D_0 \equiv 0(4)$, $D_0 \not\equiv 0(8)$. Then L has a Jordan decomposition $L = L_0 \perp {}^2L_1$ with L_0 even unimodular, L_1 odd unimodular, L_0, L_1 binary. Again fix a value $q = f(v)$. By the proof given before, there is only one orbit of vectors with $e_v = 0$ (necessarily q odd). If $e_v = 1$, then one may assume $v \in {}^2L_1$. It now follows from [Tr, Section 3] or [Kn2, §2] that the only additional invariant for conjugacy is the ideal $(v|\mathring{L}_1)$, where $\mathring{L}_1 \subseteq L_1$ is the sublattice of even vectors, and the ideal can be $2\mathbf{Z}_2$ or $4\mathbf{Z}_2$ (with respect to the original form). The vectors with $(v|\mathring{L}_1) = 4\mathbf{Z}_2$ are the called characteristic vectors in [Tr]. If L is arbitrary, and i_{min} the smallest index such that L has a $2^{i_{min}}$-modular component, similarly i_{max}, then this criterion holds for any vector v such that $v \in L_i$

for $i = i_{min}$ or $i = i_{max}$ for an appropriately chosen Jordan decomposition $L = \perp L_j$. (In the terminology of [Tr], i_{min} resp. i_{max} is the only "critical index" of v.) The ideal to be considered is of course $2^{i+1}\mathbb{Z}_2$. In order to show that there is actually only one orbit with $e_v = 1$ and given q, we have to show that $(v|\overset{\circ}{L}_1)$ is determined by q. But it is immediately seen that $(v|\overset{\circ}{L}_1) = 2\mathbb{Z}_2$ for q odd, and $(v|\overset{\circ}{L}_1) = 4\mathbb{Z}_2$ for q even.

Case 3. $D_0 \equiv 0(8)$. Then L has a Jordan decomposition $L_0 \perp {}^2L_1 \perp {}^4L_2$ with L_0, L_1, L_2 unimodular, L_1, L_2 one-dimensional. For vectors with $e_v = 0$, there is only one orbit for fixed q, as before. This also holds if $e_v = 1$ and q is odd, but the proof is different. Indeed, in this case $\mathbb{Z}_2 v$ is a possible Jordan constituent ${}^2L_1$, and there is no general theorem which guarantees the uniqueness of the orthogonal complement $L_0 \perp {}^4L_2$. In this case consider $M := 2^{-1}(L \cap 2^2 L^{\#}) + L$ which has a Jordan decomposition $M = M_0 \perp {}^2M_1$, M_0 unimodular, threedimensional, $M_1 = L_1$, and has the same orthogonal group ($O(M) \subseteq O(L)$ follows from $L = \overset{\circ}{M} = \{x \in M \mid q(x) \in \mathbb{Z}\}$). Therefore, we can replace L by M, and the previous proof applies. Notice that v is automatically characteristic, since M_1 is onedimensional.

If q is even and $e_v = 2$, then $\mathbb{Z}_2 v$ is a possible ${}^4L_2$, and precisely the same proof applies inside the original L.

It remains to show that there is only one orbit for q even, $e_v = 1$. In this case, there is no Jordan decomposition such that v is contained in one single component, and the above criterion does not apply. Again, we replace L by M and notice that $v/2 \in M$, and $v/2$ is automatically non-characteristic in M, since $(v/2|\overset{\circ}{M}) = (v/2|L) = 2^{e_v - 1}\mathbb{Z}_2 = \mathbb{Z}_2$.

For the global case, one uses the fact that $v^{\perp}$ is indefinite, of rank 3, and therefore Strong Approximation applies. For details, see e.g. [Ja, So]. The condition on the existence of enough spinor norms is fullfilled since $v^{\perp}$ has at least one Jordan component of rank ≥ 2. □

Notice that in Case 2, $D_0/4$ may be any odd squarefree number. The assumption that $-D_0$ is a fundamental discriminant, i.e. $D_0/4 \equiv 1(4)$ is not needed. In particular, Proposition 3.1 covers all forms $x_0x_1 + x_2^2 + px_3^2$, p a prime number, treated by Mennicke in [E-G-M, Theorem 11.3], and proves that the list of roots given there is indeed a complete set of representatives. For a given L of the type considered in this paper, it is always easy to write down representatives for all orbits, and therefore Proposition 3.1 can be considered as a full classification of the roots of the forms in question.

In the next proposition, we speak about fundamental roots. We therefore have to fix a generating set (possibly infinite) of $W(L)$ considered as a Coxeter group. By Vinberg's algorithm, we can make this concrete by fixing some "point" in H^3, that is, a vector $p_0 \in \mathbb{Z}^4$ such that $f(p_0) < 0$.

Proposition 3.2. Each of the following conditions on the lattice L and the value q implies that two distinct fundamental roots v, v' s.th. $f(v) = f(v') = q$ are never conjugate under $W(L)$:

i) $D_0 \neq 3$, and there exists a prime number $p \neq 2$ such that $p|q,\ p|D_0$.
ii) There exists a prime number $p \neq 2$ such that $p|q,\ p|s$, and $(\frac{3D_0}{p}) \neq 1$.
iii) $2|q,\ 2|D_0$.
iv) $D_0 = 3,\ q \not\equiv s(3)$
v) $D_0 = 4$, there exists a prime number $p \neq 2$ such that $p \mid q,\ p \equiv \pm 5\ (12)$.

Proof: Denote by R the set of fundamental roots v of L, and by $(m_{v,v'})_{v,v' \in R}$ the Coxeter matrix. From Tits' theorem about the word problem in Coxeter groups [Ti2] it follows that v, v' are conjugate under $W(L)$ only if they are in the same connected component in the graph on R with edges $\{\{v, v'\} \mid m_{v,v'} \text{ odd}\}$. Since $W(L) \subseteq GL_4(\mathbb{Z})$, the only possible odd value is $m_{vv'} = 3$. If this holds, then $f(v) = f(v') = -(v|v') = q$; that is, v, v' generate a sublattice isomorphic to the scaled root lattice ${}^qA_2$. Furthermore, $\mathbb{Z}v + \mathbb{Z}v' \subseteq qL^\#$, since v, v' are roots.

Now assume that one of the conditions i) – v) is satisfied, and furthermore assume $p \neq 3$, resp. $3 \nmid q$ in Case iv). The proof will now follow from the fact that an embedding ${}^qA_2 \hookrightarrow L \cap qL^\#$ does not exist, already over $\mathbb{Z}_p$ ($p = 2$ in Case iii), $p = 3$ in Case iv)). For instance, in Case i) or iii), the sublattice $\mathbb{Z}v + \mathbb{Z}v' \simeq {}^qA_2$ would have to be a p-Jordan component of $L\mathbb{Z}_p$. But in Case i) this component is one-dimensional since we have assumed $p|D_0$ and thus $p \nmid s,\ p^2 \nmid D$. In Case iii), the 2-Jordan component is an odd lattice, whereas ${}^qA_2$ is even. In Case ii), the quadratic congruence comes from looking at the determinant of the p-Jordan component.

To treat also the case $p = 3$, and to rule out the possibility $3|q$ in Case iv), we have to use an additional argument. Indeed, the proof given so far also shows that $m_{vv'} = 6$ is impossible, but this value actually occurs for certain forms in Cases i), ii), $p = 3$, and Case iv), $3|q$. To prove that $m_{vv'} \neq 3$ also in these remaining cases, one shows that $v - v' \in 3qL^\#$, that is, $v - v'$also is a root. This now easily implies that one of the roots v, v' is not fundamental, a contradiction. □

The following Proposition, which is an easy consequence of the last two Propositions, allows to prove "in practice" the infiniteness of $[O(L) : W(L)]$ for lattices L where the criteria of § 2 do not apply.

Proposition 3.3. Suppose that L and q satisfy one of the conditions stated in Proposition 3.2. Let $o(L, q) = 1, 2$ be the number of orbits of $O^+(L)$ on roots v with $f(v) = q$. If f has more than $4 \cdot o(L, q)$ fundamental roots v with $f(v) = q$, then $W(L)$ is of infinite index in $O^+(L)$.

Proof. The stabilizer $A = A(L)$ in $O^+(L)$ of any fundamental domain for $W(L)$ is a complement to $W(L)$ in $O^+(L)$. If A is finite, it obviously follows from Proposition 3.2 that the number of fundmental roots with $f(v) = q$ is at most $|A| \cdot o(L,q)$. The bound $|A| \leq 4$ follows from the following Proposition, taking into account the fact that a finite A is isomorphic to a subgroup of $SL_3(\mathbf{Z})$. □

Once Vinberg's algorithm is implemented on a computer for our particular forms $\mathbf{H} \perp {}^s[a,b,c]$, it is now a tedious, but straightforward verification that $[O(L) : W(L)] = \infty$ in all those $324-(49+2)$ cases of the list derived in § 2 where Vinberg's algorithm does not stop with a fundamental domain of finite volume. In fact, there are only very few cases, like $[1,1,9]$, $D = 35$, where a nontrivial computational effort is needed. Usually, the fundamental domain is easily seen to be finite, or its number of faces within fixed distance C from the base point turns to infinity very rapidly with C.

We conclude this paper by a general result on rotations of finite order in integral orthogonal groups of signature $(3,1)$. This seems to be of some geometrical and group theoretical interest, independently of our particular theme of classifying reflective lattices. The following Proposition has been suggested by an unpublished result of Mennicke's [Men2], who treated rotations of order 3 for lattices $L = \mathbf{H} \perp [1,0,d]$, d squarefree. In the course of the proof of our main Theorem, the result is only used as a technical Lemma which improves the trivial bound $|A| \leq 24$ (the group A as defined in the proof of Proposition 3.3) to $|A| \leq 4$, thereby leading to substantial numerical simplifications in the application of Proposition 3.3.

Proposition 3.4. Any rotation of order 3 or 4 in $O^+(L)$ is a product of two reflections in $O^+(L)$. Here, L can be any integral lattice of signature $(3,1)$.

Proof: The reader may consult [Qu] and some of the papers quoted there for the general theory behind the following proof. Let $\varphi \in SO^+(L)$ be of order 3. Let ω be a third root of unity, $K = \mathbf{Q}(\omega)$, $R = \mathbf{Z}[\omega]$. The vectorspace $V = \mathbf{Q}L$ has a decomposition $V = V_1 \oplus V_\omega$, where $V_1 = ker(\varphi - 1)$, $V_\omega = ker(\varphi^2 + \varphi + 1)$, and V_ω is a vectorspace over K by letting ω act like φ. The constituents V_1, V_ω are automatically orthogonal to each other. The scalar product on V_ω is of the form $tr \circ h$, where $tr : K \to \mathbf{Q}$ is the trace, and $h : V_\omega \times V_\omega \to K$ is a hermitian form. Since the signature on the whole of V is $(3,1)$, and $\varphi \neq 1$, necessarily $dim V_1 = dim_{\mathbf{Q}} V_\omega = 2$. Consider the embedding $L \subseteq L_1 \perp L_\omega$, where L_1, L_ω denote the projections onto V_1, V_ω, and the related embedding of the rings

$$\mathbf{Z}[\varphi] \cong \mathbf{Z}[x]/(x^3 - 1) \to \mathbf{Z} \times R, \quad \varphi \mapsto (1, \omega).$$

Since $\varphi - 1 \mapsto (0, \omega - 1)$ and $1 + \varphi + \varphi^2 \mapsto (3,0)$, the image contains $3\mathbf{Z} \times \pi R$, where $\pi = \omega - 1$ is a prime element in R, $R/\pi R \cong \mathbf{Z}/3\mathbf{Z} = \mathbf{F}_3$. From this it follows immediately that $3L_1 \perp \pi L_\omega \subseteq L$. The factor module $L/(3L_1 \perp \pi L_\omega) =: \bar{L}$ is of dimension 3 or 2 over F_2, since $L_1/3L_1 \perp L_\omega/\pi L_\omega$ has dimension 3 over F_3, and the projection of $\bar{L}$ onto the first factor is surjective. Therefore, L is of the form

$$L = \{(x, y) \in L_1 \times R \mid \ell(x) + \varepsilon y \in \pi R\},$$

for some linear form $\ell : L_1 \to \mathbf{Z} \hookrightarrow R$, and some $\varepsilon \in R$. The involution σ which is defined as the identity on V_1, and complex conjugation on V_ω, is an isometry of V, and in fact $\varphi = \sigma \circ (\sigma \circ \varphi)$ is a representation of φ as a product of two +-reflections. The only thing left to show is that σ maps L into itself. This is clear from the above description of L, since complex conjugation acts trivially on $R/\pi R$.

If φ is of order 4, then, after possibly replacing φ by $-\varphi$, we may assume that -1 is not an eigenvalue of φ. Now exactly the same proof as above applies, replacing $x^3 - 1$ by $(x-1)(x^2+1)$, ω by $i = \sqrt{-1}$, and 3 by 2 at various places, e.g. $R/\pi R \cong \mathsf{F}_2$, $2L_1 \perp \pi L_i \subseteq L$. □

Table. The fundamental polytopes of the maximal reflection groups $W(L)$.

<u>Notation</u>

$L = \mathsf{H}\perp^{s}[a,b,c] = (\mathbb{Z}^4, f)$, where $f = x_0x_1 + s(ax_2^2 + bx_2x_3 + cx_3^2)$

$D = s^2(4ac - b^2)$ the discriminant

$D_0 = 4ac - b^2$ the fundamental discriminant

v = number of vertices including cusps

e = number of edges

r = number of faces = number of fundamental roots

r_i = number of faces with i vertices

c = number of cusps

i = index $[GO^+(L) : W(f)]$, $GO^+(L)$ the group of similarities of L

= order of the automorphism group of the fundamental polytope

No.	$^s[a,b,c]$	D	D_0	v	e	r	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}	r_{12}	c	i
1.	[1,1,1]	3	3	4	6	4	4									1	1
2.	[1,0,1]	4	4	4	6	4	4									1	1
3.	[1,1,2]	7	7	7	11	6	2	4								1	2
4.	[1,0,2]	8	8	5	8	5	4	1								1	1
5.	[1,1,3]	11	11	7	11	6	2	4								1	2
6.	$^2[1,1,1]$	12	3	4	6	4	4									1	1
7.	[1,1,4]	15	15	10	16	8	4		4							2	4
8.	[2,1,2]	15	15	7	11	6	2	4								1	2
9.	[1,1,5]	19	19	9	14	7	2	3	2							1	2
10.	[1,0,5]	20	20	7	11	6	2	4								1	1
11.	[2,2,3]	20	20	7	11	6	2	4								1	2
12.	[1,0,6]	24	24	7	11	6	2	4								1	1
13.	[2,0,3]	24	24	7	11	6	3	2	1							1	1
14.	$^2[1,1,2]$	28	7	10	16	8	4		4							2	2
15.	[3,1,3]	35	35	11	17	8		6	2							1	2
16.	$^3[1,0,1]$	36	4	6	9	5	2	3								1	1
17.	[1,1,10]	39	39	14	22	10	4	2	2		2					2	2
18.	[1,0,10]	40	40	12	19	9	4	2		3						2	2
19.	[2,0,5]	40	40	9	14	7	1	5	1							1	1
20.	$^2[1,1,3]$	44	11	9	14	7	2	3	2							1	2
21.	[1,0,13]	52	52	14	22	10	4		4	2						2	1
22.	[1,0,14]	56	56	12	19	9	3	2	3	1						2	1
23.	$^2[1,1,4]$	60	15	14	22	10	2	4	2	2						2	2

No.	${}^s[a,b,c]$	D	D_0	v	e	r	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}	r_{12}	c	i
24.	${}^3[1,1,2]$	63	7	14	22	10	4		4	2						2	2
25.	[1,0,17]	68	68	19	30	13	6	2		2	2	1				3	2
26.	${}^3[1,0,2]$	72	8	10	16	8	2	4	2							2	2
27.	${}^5[1,1,1]$	75	3	8	12	6		6								2	2
28.	[1,0,21]	84	84	16	25	11	4	2	2	1	2					2	
29.	[3,0,7]	84	84	11	17	8	2	3	2	1						1	1
30.	[5,4,5]	84	84	17	26	11		4	6	1						1	
31.	${}^5[1,0,1]$	100	4	8	12	6		6								2	2
32.	[1,0,30]	120	120	17	26	11	3	2	3	2		1				1	
33.	[2,0,15]	120	120	22	34	14	2	4	2	6						2	
34.	[3,0,10]	120	120	20	31	13	2	6	2		2	1				2	
35.	[5,0,6]	120	120	22	34	14		8	2	2	2					2	
36.	[1,0,33]	132	132	24	37	15	4	4	2	2	2			1		2	
37.	${}^2[3,1,3]$	140	35	17	26	11		4	6	1						1	
38.	${}^7[1,1,1]$	147	3	8	12	6		6								2	2
39.	[3,0,14]	168	168	30	46	18	4	2	6		6					2	2
40.	[6,0,7]	168	168	30	46	18		6	8	1	2	1				2	2
41.	${}^3[1,0,5]$	180	20	23	36	15	2	6		7						3	
42.	${}^7[1,0,1]$	196	4	14	21	9		3	6							2	2
43.	${}^5[1,0,2]$	200	8	20	31	13		8		5						2	
44.	${}^{10}[1,1,1]$	300	3	10	15	7		5	2							1	1
45.	${}^3[1,0,10]$	360	40	34	52	20	4	2	6	6			2			2	
46.	${}^3[2,0,5]$	360	40	34	52	20		10	4	3	2				1	2	
47.	${}^{13}[1,1,1]$	507	3	20	30	12		4	4	4						4	
48.	${}^{14}[1,1,1]$	588	3	20	30	12		4	4	4						4	2
49.	${}^{15}[1,0,1]$	900	4	32	48	18		4	4	10						4	

Literature

[Bo-Ha] A. Borel and Harish-Chandra: Arithmetic subgroups of algebraic groups. Ann. of Math. **75** (1962), 485–535.

[Bo] A. Borel: Commensurability classes and volumes of hyperbolic 3-manifolds. Ann. Sc. Norm. Super. Pisa, Cl. Sci., (4) **8** (1981), 1–33.

[B-S] Z. I. Borevich and I. R. Shafarevich: *Number Theory.* Academic Press New York, San Francisco, London, 1966.

[B-T] F. Bruhat and J. Tits: Schémas en groupes et immeubles des groupes classiques sur un corps local. Deuxième partie: Groupes unitaires. Bull Soc. math. France **115** (1987), 141–195.

[Bu1] V. O. Bugaenko: Groups of Automorphisms of Unimodular Hyperbolic Quadratic Forms over the Ring $Z\left[\frac{\sqrt{5}+1}{2}\right]$. Vestnik Moskovskogo Universiteta, Matematika, **39** (1984), 6–12.

[Bu2] V. O. Bugaenko: On Reflective Unimodular Hyperbolic Quadratic Forms. *Problems in Group Theory and Homological Algebra*, A. L. Oniščik (ed.), Jaroslavl 1987.

[C-S] J. H. Conway, N. J. A. Sloane: *Sphere Packings, Lattices and Groups.* Springer-Verlag New York, Berlin, Heidelberg, 1988.

[E-G-M] J. Elstrodt, F. Grunewald, and J. Mennicke: Discontinuous Groups on Three-dimensional Hyperbolic Space: Analytical Theory and Arithmetic Applications. Russian Math. Surveys **38** (1983), 137–168.

[He] H. Helling: On the Commensurability Class of the Rational Modular Group. J. London Math. Soc. (2), **2** (1970), 67–72.

[Hs1] J. S. Hsia: An invariant for integral equivalence. Am. J. Math. **93** (1971), 867–871.

[Hs2] J. S. Hsia: Integral Equivalence of Vectors over Depleted Modular Lattices on Dyadic Local Fields.

[Ja] D. G. James: Representations by Integral Quadratic Forms. J. Number Theory **4** (1972), 321–329.

[Ka] I. M. Kaplinskaya: Discrete Groups Generated by Reflections in the Faces of Simplicial Prisms in Lobachevskian Spaces. Matematicheskie Zametki **15** (1974), 159–164.

[Kn1] M. Kneser: Starke Approximation in algebraischen Gruppen. I. J. reine angew. Math. **218** (1965), 190–203.

[Kn2] M. Kneser: Quadratische Formen. Vorlesungsausarbeitung, Göttingen, 1974.

[Kn3] M. Kneser: Witts Satz für quadratische Formen über lokalen Ringen.

Nachr. Akad. Wiss. Göttingen, Math.-Phys. Klasse (1972), 195–203.

[Kn4] M. Kneser: Erzeugung ganzzahliger orthogonaler Gruppen durch Spiegelungen.
Math. Ann. **255** (1981), 453–462.

[Ko] J. L. Koszul: Lectures on Hyperbolic Coxeter Groups.
University of Notre Dame, 1967.

[La] F. Lanner: On Complexes with Transitive Groups of Automorphisms.
Lunds Univ. Math. Sem. **11** (1950).

[Mac] C. Maclachlan: Groups of Units of Zero Ternary Quadratic Forms.
Proc. Royal Soc. Edinburgh **88A** (1981), 141–157.

[Ma1] V. S. Makarov: A class of partitions of a Lobachevskii space.
Soviet Math. Dokl. **6** (1965), 400–401.

[Ma2] V. S. Makarov: A class of discrete groups in Lobachevskii space having an infinite fundamental domain of finite measure.
Soviet Math. Dokl. **7** (1977), 328–331.

[Men1] J. Mennicke: Eine Pflasterung des dreidimensionalen hyperbolischen Raumes.
Math. Phys. Semesterberichte **XXVII** (1980), Heft 1, 55–68.

[Men2] Unpublished manuscript, Bielefeld 1982.

[Mey] J. Meyer: Präsentation der Einheitengruppe der quadratischen Form $F(X) = -X_0^2 + X_1^2 + \cdots + X_{18}^2$.
Archiv Math. **29** (1977), 261–266.

[M-T] G. D. Mostow, T. Tamagawa: On the compactness of arithmetically defined homogeneous spaces.
Ann. of Math. **76** (1962), 440–463.

[N-S] A. Neumaier, J. J. Seidel: Discrete hyperbolic geometry.
Combinatorica **3** (1983), 219–237.

[Ni1] V. V. Nikulin: On Factor Groups of the Automorphism Groups of Hyperbolic Forms modulo Subgroups Generated by 2-Reflections.
Soviet Math. Dokl. **20** (1979), 1156–1158.

[Ni2] V. V. Nikulin: On Arithmetic Groups Generated by Reflections in Lobachevsky Spaces.
Math. USSR Izvestiya **16** (1981), 573–601.

[Ni3] V. V. Nikulin: On the Classification of Arithmetic Groups Generated by Reflections in Lobachevsky Spaces.
Math. USSR Izvestija **18** (1982), 99–123.

[OM] O. T. O'Meara: *Introduction to Quadratic Forms.*
Second printing, Springer Verlag, Berlin, Heidelberg, New York 1971.

[Pr] M. N. Prokhorov: The Absence of Discrete Reflection Groups with Non-compact Fundamental Polyhedron of Finite Volume in Lobachevsky Space of Larges Dimension.
Math. USSR Izvestiya **28** (1987), 401–411.

[Qu] H. G. Quebbemann: Zur Klassifikation unimodularer Gitter mit Isometrie von Primzahlordnung.
J. reine angew. Math. **326** (1981), 158–170.

[Ša] M. K. Šaiheyev: Reflection Subgroups in Bianchi groups (in Russian).
Problems in Group Theory and Homological Algebra, A. L. Oniščik (ed.), Jaroslavl 1987.

[Ro1] J. Rohlfs: Über maximale arithmetisch definierte Gruppen.
Math. Ann. **234** (1978), 239–252.

[Ro2] J. Rohlfs: Die maximalen arithmetisch definierten Untergruppen zerfallender einfacher Gruppen.
Math. Ann. **244** (1979), 219–231.

[So] K.-P. Sondergeld: Witt's Theorem for Square-free Lattices over Dyadic Discrete Valuation Rings.
Arch. Math. **43** (1984), 510–514.

[Šv] O. V. Švarcman: Reflection Subgroups in Bianchi Groups (in Russian).
Problems in Group Theory and Homological Algebra, A. L. Oniščik (ed.), Jaroslavl 1987.

[Ti1] J. Tits: Groupes et géométries de Coxeter.
I.H.E.S., Paris 1961.

[Ti2] J. Tits: Le problème des mots dans les groupes de Coxeter.
Istituto Nazionale die Alta Matematica, Symposia Mathematica **I** (1968), 175–185.

[Ti3] J. Tits: Reductive groups over local fields.
Proc. Symp. Pure Math. **33** (1979), Summer Inst. on Group representations and automorphic forms, Corvalis 1977, vol. 1, 29–69.

[Tr] A. Trojan: The Integral Extension of Isometries of Quadratic Forms over Local Fields.
Canadian J. of Math. **18** (1966), 920–942.

[Vi1] E. B. Vinberg: Discrete Groups Generated by Reflections in Lobačevskiĭ Spaces.
Math. USSR Sbornik **1** (1967), 429–444.

[Vi2] E. B. Vinberg: Some Examples of Crystallographic Groups in Lobačevskiĭ Spaces.
Math. USSR Sbornik **7** (1969), 617–622.

[Vi3] E. B. Vinberg: On Unimodular Integral Quadratic Forms.
Functional Anal. Appl. **6** (1972), 24–31.

[Vi4] E. B. Vinberg: On Groups of Unit Elements of Certain Quadratic Forms.

Math. USSR Sbornik **16** (1972), 17–35.

[Vi5] E. B. Vinberg: Some Arithmetical Discrete Groups in Lobačevskiĭ Spaces. *Discrete subgroups of Lie groups*, Oxford Univ. Press 1975, 323–348.

[Vi6] E. B. Vinberg: The Absence of Crystallographic Groups of Reflections in Lobachevsky Spaces of Large Dimension.
Trans. Moscow Math. Soc. **47** (1985), 75–112.

[Vi7] E. B. Vinberg: Reflection Subgroups in Bianchi Groups (in Russian).
Problems in Group Theory and Homological Algebra, A. L. Oniščik (ed.), Jaroslavl 1987.

[Vi-Ka] E. B. Vinberg and I. M. Kaplinskaja: On the Groups $O_{18,1}(Z)$ and $O_{19,1}(Z)$.
Soviet Math. Dokl. **19** (1978), 194–197.

[Wa1] G. L. Watson: Transformations of a quadratic form which do not increase the class-number.
Proc. London Math. Soc. **12** (1962), 577–587.

[Wa2] G. L. Watson: Transformations of a quadratic form which do not increase the class-number (II).
Acta Arithmetica **XXVII** (1975), 171–189.

[Za] D. B. Zagier: *Zeta-Funktionen und quadratische Körper.*
Springer Verlag Berlin, Heidelberg, New York 1981.