

# Classifying lattices using modular forms – a preliminary report

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## 1 Introduction and main results

For some years now, computer programs are available ([SchH] and MAGMA) which generate representatives for all isometry classes of lattices (integral quadratic forms) in a genus of dimension say up to 16 and not too large determinant. In some cases, when all lattices turn out to have a rich structure, it is desirable to have a better insight and a computer-free alternative to the classification. This is in particular possible if the lattices  $L$  have a large root system  $\mathbf{R}(L)$  (as introduced in [SchBl]). For instance, if  $L$  is even and of prime level  $\ell$ , then there are short and long roots of norms 2 and  $2\ell$ , respectively. The long roots correspond to the short roots of the rescaled dual lattice  $L^\natural$  (see below for unexplained notions). Now one makes use of the classical fact that the theta series of  $L$  and  $L^\natural$  are transformed into each other, up to a factor, by the action of the Fricke involution. Analysing the formulae leads one to introduce a modification of the Coxeter number of a root system. Using the action of the Fricke involution on theta series with harmonic coefficients, one can show for small  $\ell$  and appropriate dimensions, that this modified Coxeter number is the same for all irreducible components of the root system  $\mathbf{R} = \mathbf{R}(L)$ . In this way, one can determine all possibilities for the root system.

By ‘lattice’ we mean as usual a  $\mathbb{Z}$ -lattice  $L$  of full rank in a rational vector space with a positive definite scalar product  $(x, y)$ . It is called *integral* if  $(x, y) \in \mathbb{Z}$  for all  $x, y \in L$ , and *even*, if  $(x, x) \in 2\mathbb{Z}$  for all  $x \in L$ . Any even lattice is integral. A lattice is called *p-elementary*, for some prime number  $p$ , if  $pL^\# \subseteq L$ , where  $L^\# = \{y \in \mathbb{Q}L \mid (x, y) \in \mathbb{Z} \text{ for all } x \in L\}$  denotes the *dual lattice* of  $L$ .

By a left upper index  $\alpha \in \mathbb{Q}^+$ , we denote scaling:  ${}^\alpha L$  equals  $L$  as a lattice, but the form is multiplied by  $\alpha$ . If  $L$  is  $p$ -elementary, then  ${}^p(L^\#)$  is again (integral and)  $p$ -elementary. We write  ${}^p(L^\#) = L^\natural$  for short. For an even lattice  $L$ , the smallest number  $\ell$  s.t.  ${}^\ell(L^\#)$  is again even is called the *level* of  $L$ . If  $\ell$  is a prime number  $\neq 2$ , an even lattice  $L$  has level  $\ell$  if and only if  $L$  is  $\ell$ -elementary and  $L \neq L^\#$ . An even lattice has level 2 if and only if it is 2-elementary,  $L \neq L^\#$ , and  $L$  is *totally even* in the sense that both  $L$  and  ${}^2L^\#$  are even. If  $L$  has level  $\ell$  (prime

or not), then its theta series is of level  $\ell$ , i.e. a modular form for the congruence subgroup  $\Gamma_0(\ell)$  of  $\mathrm{SL}_2(\mathbb{Z})$ ; see e.g. [Miy].

The lattices of level 1 are the well-studied even unimodular lattices; the lattices of level 3 and dimension at most 12 or of level 2 and dimension at most 16 are the proper object of investigation in this paper. From the point of view of concrete classifications of lattices, the following three theorems are the main results of this paper.

Before stating the results, we recall from [SchBl] the definition

$$R(L) := \{v \in L \mid v \text{ primitive, } 2(v, L)/(v, v) \subseteq \mathbb{Z}\}$$

of the *root system* of an arbitrary lattice  $L$ . The quadratic form is part of the structure of such root systems  $R$ ; therefore, scaling  ${}^\alpha R$  makes sense. The lattice generated by a root system  $A, B, C, \dots$  is denoted by  $A, B, C, \dots$ . The root system of a lattice is (when neglecting the scaling) a root system in the usual sense of Lie algebra theory as treated in [Bou]; in particular it decomposes uniquely into indecomposable components with the standard names  $A_r, B_r, \dots, G_2$ . Specializing again to lattices of prime level  $\ell$ , we observe, that

$$R(L) = \{v \in L \mid (v, v) = 2\} \dot{\cup} \{v \in \ell L^\# \mid (v, v) = 2\ell\} =: R_{sh} \dot{\cup} R_{lo}$$

(‘short’ and ‘long’ roots). Thus, the possible irreducible components of  $R(L)$  are  $A_r, {}^\ell A_r, D_r, {}^\ell D_r, E_6, {}^\ell E_6, E_7, {}^\ell E_7, E_8, {}^\ell E_8$  and the mixed types  ${}^2 B_r, C_r$  ( $\ell = 2$ ) and  $G_2$  ( $\ell = 3$ ).

**Theorem 1** *There are 10 isometry classes of lattices of rank 12, level 3 and determinant  $3^6$ . Representatives are the Coxeter-Todd lattice and 9 lattices having the following root systems:*

$$\begin{array}{cccccc} 6A_1 & 6{}^3A_1 & 3A_2 & 3{}^3A_2 & 2A_3 & 2{}^3A_3 & 6G_2 & D_4 & {}^3D_4 & 2G_2 \\ A_5 & {}^3A_5 & G_2 & A_6 & {}^3A_6 & D_6 & {}^3D_6 & E_6 & {}^3E_6 \end{array}$$

Notice that all but exactly one of these lattices are *reflective* in the sense that the root system has maximal rank 12. This is completely analogous to the famous case of 24-dimensional, even unimodular lattices [Ven], or 16-dimensional lattices of level 2 and determinant  $2^8$  [SchV] (genus of  $4D_4$  or the Barnes-Wall lattice).

**Theorem 2** *There are  $17 + 11 + 6$  isometry classes of lattices of rank 16, level 2 and determinant  $2^6, 2^4, 2^2$ , respectively. They are all reflective and have the root systems shown in Tables 2, 3 and 4 below, and are uniquely determined by their respective root system.*

**Theorem 3** *There are  $6 + 3$  isometry classes of lattices of rank 12, level 3 and determinant  $3^4, 3^2$ , respectively. They are all reflective, have the root systems*

$$\begin{array}{cccccc} 6A_2 & A_5 & D_4 & 3{}^3A_1 & A_8 & 2{}^3A_2 & D_7 & {}^3A_3 & G_2 & E_6 & 3G_2 & E_7 & {}^3A_5 \\ 2E_6 & D_{10} & 2{}^3A_1 & E_8 & 2G_2 \end{array}$$

*and are uniquely determined by their respective root system.*

## 2 Lattices of level 3: the root systems

Our aim is the classification of lattices of level 3 in dimensions  $n$  at most 12. In view of the unique decomposition of an arbitrary (positive definite) lattice as an orthogonal sum of indecomposable ones, which are necessarily also of level 1 or 3, the classification immediately reduces to the case  $n = 12$ . This is covered by Theorems 1 and 3 above. In the following, we present a detailed proof of Theorem 1, which was announced in [SchV] and deals with the most difficult case. We will not treat Theorem 3 in this paper, which can be obtained along the lines of Theorem 2, but with fewer computations.

For a given dimension  $n$  and determinant  $\det L = |L^\# / L| = 3^r$ , all 3-elementary even lattices form one genus. They exist if and only if either  $r \in \{0, n\}$  and  $n \equiv 0(8)$ , or  $0 < r < n$ ,  $2r \equiv n \pmod{4}$ . If  $n = 2m$ , the mapping  $L \mapsto L^\natural$  is an involution on (the isometry classes of) this genus. A  $\natural$ -invariant representative (i.e. a ‘modular’ lattice in the sense of [Que]) is  $mA_2 = A_2 \perp \dots \perp A_2$ , where  $A_2$  as usual denotes the  $A_2$ -root lattice.

In the case of Theorem 1 ( $n = 12, r = 6$ ) we shall call this genus  $\mathcal{H}$  for short. Another, prominent, lattice in  $\mathcal{H}$  is the Coxeter-Todd lattice  $C$  (see [CT]) which has no vector of norm 2 and again satisfies  $C \cong C^\natural$ . It is in fact a consequence of our classification that this holds for all lattices in  $\mathcal{H}$ .

Following [SchV] we now derive a list of ‘possible’ root systems of lattices in  $\mathcal{H}$  which are subject to two essential restrictions. In the next section we shall show that for each root system in our list, there exists a unique lattice.

As usual,  $h$  denotes the Coxeter number of an irreducible root system  $R$ . It is characterized by the following formula:

$$(1) \quad \sum_{v \in R} \frac{(v, x)^2}{(v, v)} = h \cdot (x, x) \text{ for all } x \in \mathbb{QR}.$$

See e.g. [Bou], Chap.V, §6.2, p. 121.

We now collect the properties of the genus  $\mathcal{H}$ , and of the Coxeter-Todd lattice, which will be used in the sequel.

**Lemma 4** *The theta-series of an arbitrary lattice  $K$  in  $\mathcal{H}$  coincides with the theta-series of its 3-scaled dual lattice:  $\theta_K = \theta_{K^\natural}$ .*

**Proof:** This is essentially contained in [Que], we briefly recall the argument. The theta-series  $\theta_K$  is a modular form of weight 6 for the congruence subgroup  $\Gamma_0(3) \subset SL_2(\mathbb{Z})$ . The Fricke involution  $t_3 \in SL_2(\mathbb{Z})$  normalizes  $\Gamma_0(3)$  and thus acts on the space of those modular forms as well as on the subspace of cusp forms. By definition,  $t_3$  interchanges  $\theta_K$  and  $\theta_{K^\natural}$ . Furthermore, the difference of any two  $\theta$ -series of lattices in  $\mathcal{H}$ , in particular  $\theta_K - \theta_{K^\natural}$ , is a cusp form. The special fact now is that the space of cusp forms of weight 6 is one-dimensional; see [Miy], Table A, p. 296. Thus  $t_3$  fixes every cusp form, and consequently  $\theta_K = \theta_{K^\natural}$ .

Let  $K$  be any lattice in  $\mathcal{H}$ , and  $T(K) := K^\# / K$  its discriminant group. The induced bilinear form  $T(K) \times T(K) \rightarrow \mathbb{Q}/\mathbb{Z}$  (discriminant form) is regular and takes its values in  $\mathbb{Z} \cdot \frac{1}{3}$ . After multiplication by 3, it can be considered as an ordinary inner product on the  $\mathbb{F}_3$ -vector space  $T(K)$ . Clearly, this inner product is of determinant 1 (look at  $K = 6A_2$ ), and thus isometric to  $\mathbb{H} \perp \mathbb{H} \perp \langle 1, 1 \rangle$ , where  $\mathbb{H}$  denotes a hyperbolic plane over  $\mathbb{F}_3$  and  $\langle 1, 1 \rangle$  a two-dimensional space with orthonormal basis. The number of elements of norm  $\alpha \neq 0 \in \mathbb{F}_3$  in such a space is known to be  $q^5 + q^2$  with  $q = 3$ , i.e. 252.

Combining these general properties of  $\mathcal{H}$  with the special properties of the Coxeter-Todd lattice  $C$ , we arrive at the following lemma.

**Lemma 5** *Let  $C$  be the Coxeter-Todd lattice and  $T = T(C)$  its discriminant group, equipped with the discriminant bilinear form.*

- a) *The orthogonal group  $O(C)$  maps surjectively onto the orthogonal group  $O(T)$ .*
- b) *The minimal norm of  $C^\#$  is  $\frac{4}{3}$ . Every anisotropic element in  $T$  has precisely three representatives of norm  $\frac{4}{3}$ .*

For the proof of a), one has to invoke one of the explicit constructions of  $C$ , see e.g. [CT]. Part b) is then derived as follows: The minimal norm  $\frac{4}{3}$  of  $C^\#$  follows from Lemma 4, using the basic fact that  $C$  itself has minimum 4. It is known from either the explicit construction, or from modular forms theory, that there are  $756 = 3 \cdot 252$  minimal vectors. Since the orthogonal group of  $T$  acts transitively on the elements of norm  $\frac{1}{3} + \mathbb{Z}$ , it follows from a) that the  $3 \cdot 252$  vectors of norm  $\frac{4}{3}$  in  $C^\#$  are equidistributed modulo  $C$ . The second claim of part b) follows.

We shall use the standard glueing construction of unimodular lattices  $\tilde{K} = K_1 \times_\varphi K_2 = \{(x, y) \in K_1^\# \times K_2^\# \mid \varphi \bar{x} = \bar{y}\}$  where  $\varphi : T(K_1) \rightarrow T(K_2)$  is a bijection (see Proposition 2 of [SchV]) in the following special case:  $K_1$  is an arbitrary lattice in the genus  $\mathcal{H}$ , and  $K_2 = C$ . In this case, the glueing map  $\varphi$  exists. (By Lemma 5, part a), it is even essentially unique, i.e., two possible  $\varphi$ 's can be transformed into each other by an automorphism of  $K_1 \times K_2$  preserving the components; this will not be needed in the sequel.) Thus, to any  $K \in \mathcal{H}$  we have an associated Niemeier lattice  $\tilde{K} := K \times_\varphi C$ .

An essential step to the proof of Theorem 1 is provided by the following lemma.

**Lemma 6** *Let  $K$  be a lattice in  $\mathcal{H}$  and  $R = R(K)$  its root system. Then the following properties hold:*

- a)  $R = \emptyset$  or  $\text{rank } R = 12$ .
- b) *If  $R \neq \emptyset$ , then*
  - (i) *the Coxeter number  $h$  is the same for all irreducible components of  $R$ ,*

$$(ii) \quad |\mathbf{R}_{sh}| = |\mathbf{R}_{lo}|,$$

$$(iii) \quad |\mathbf{R}_{sh}| = 6h.$$

**Proof:** Property (iii) is readily derived from properties (i) and (ii), but will also be obtained in the course of the proof of (i). The equality (ii)  $|\mathbf{R}_{sh}| = |\mathbf{R}_{lo}|$  follows immediately from Lemma 4 and the basic observation that the short roots of  $K^\#$  (or  $K^\natural$ ) are in one-to-one correspondence with the long roots of  $K$ , by mapping  $v \mapsto 3v$ .

Now assume that  $\mathbf{R} = \mathbf{R}(K) \neq \emptyset$ . Then also  $\mathbf{R}_{sh} \neq \emptyset$ , and thus  $\tilde{\mathbf{R}} := \mathbf{R}(\tilde{K}) \neq \emptyset$ . Let us describe  $\tilde{\mathbf{R}} = \{v \in \tilde{K} \mid (v, v) = 2\}$  explicitly in terms of  $K$ . The vectors  $\tilde{v}$  in  $\tilde{K}$  of norm 2 are of two kinds: either  $\tilde{v} \in K$ , or  $\tilde{v}$  is of the form  $\tilde{v} = v_1 + v_2$ ,  $v_1 \in K^\#$ ,  $v_2 \in C^\#$ ,  $(v_1, v_1) = \frac{2}{3}$ ,  $(v_2, v_2) = \frac{4}{3}$ ,  $\varphi \bar{v}_1 = \bar{v}_2$ .

Now we use Lemma 5: for every element in  $C^\#/C$  which is anisotropic with respect to the quadratic form  $3(x, x) + 3\mathbb{Z}$ , there exist exactly 3 representatives in  $C$  of norm  $(x, x) = \frac{4}{3}$ . It follows that the roots of  $\tilde{K}$  of the second kind are of the shape  $v_1 + v_2$ , where  $3v_1$  runs through the long roots of  $K$ , and to every  $v_1$  there are exactly three  $v_2$ . We see that

$$(2) \quad |\tilde{\mathbf{R}}| = |\mathbf{R}_{sh}| + 3|\mathbf{R}_{lo}| = 4|\mathbf{R}_{sh}|.$$

Now we use the following formula from [Ven], Proposition 1:

$$(3) \quad \sum_{v \in \tilde{\mathbf{R}}} (v, x)^2 = \frac{1}{12}(x, x)|\tilde{\mathbf{R}}| \text{ for all } x \in \tilde{K}.$$

The above description of  $\tilde{\mathbf{R}}$  gives

$$(4) \quad \sum_{v \in \tilde{\mathbf{R}}} (v, x)^2 = \sum_{v \in \mathbf{R}_{sh}} (v, x)^2 + 3 \sum_{v \in \mathbf{R}_{lo}} \left(\frac{v}{3}, x\right)^2 = 2 \sum_{v \in \mathbf{R}} \frac{(v, x)^2}{(v, v)} \text{ for all } x \in \tilde{K}.$$

Now specify an irreducible component  $\mathbf{R}_i$  of  $\mathbf{R}$  and choose  $x$  non-zero in  $\mathbf{R}_i$ . Substituting (4) into (3) and comparing with (1) gives

$$h_i = \frac{1}{24}|\tilde{\mathbf{R}}|$$

for the Coxeter number  $h_i$  of  $\mathbf{R}_i$ . This proves (i). (In view of (2), it also proves that  $|\mathbf{R}_{sh}| = 6h$ , i.e. the additional property (iii).)

Recall that the Coxeter numbers of the irreducible root systems are as follows:

$$h(\mathbf{A}_l) = l + 1, \quad h(\mathbf{D}_l) = 2(l - 1), \quad h(\mathbf{G}_2) = 6, \quad h(\mathbf{E}_6) = 12 < h(\mathbf{E}_7) < h(\mathbf{E}_8)$$

Having this in mind, it is easily verified that the 12-dimensional root systems consisting of roots of norm 2 and 6 only, and having properties (i) and (ii), are precisely the root systems listed in the theorem.

### 3 Constructing lattices of level 3

In this section, we will complete the proof of Theorem 1 by constructing, for each root system  $R$  of that list, a lattice  $K$  in  $\mathcal{H}$  with  $R(K) = R$ . Unlike in the case  $n = 16$ ,  $\ell = 2$  [SchV] where most of the constructions were omitted, we give a full treatment here. For each possible root system  $R$ , we specify the “glue code”  $\mathcal{K} (= K/\langle R \rangle)$ , generating the desired lattice  $K$  over its sublattice  $\langle R \rangle$ , verify its uniqueness (up to isomorphism), and determine “by hand” the automorphism group of  $\mathcal{K}$ , and thus the orthogonal group of  $K$ . The uniqueness of the code and thus of the lattice also follows from the mass formula.

We adopt the following notational conventions. A *glue vector* of a component  $X$  of  $R$  is a vector in  $\langle X \rangle^\#$  with minimal norm in its coset modulo  $\langle X \rangle$ . The relevant factor group is denoted by  $T(X) := \langle X \rangle^\# / \langle X \rangle$ , the discriminant group of  $X$ ; similarly  $T(R)$ . If  $R(K)$  contains a component  $G_2$ , then the sublattice  $A_2 = \langle G_2 \rangle \subset K$  splits off, by Proposition 2.10 of [SchBl]. Therefore, all root systems to be considered are essentially of the form  $R = Y^3Y$ , where  $Y$  is unscaled, i.e. exists of vectors of norm 2. We shall refer to  $T(Y) \times \{0\}$  and  $\{0\} \times T({}^3Y)$  as the first and second factor of  $T(R)$ . Glue vectors of  $Y$  will be denoted by  $g_1, g_2, \dots$ , those of  ${}^3Y$  by  $g'_1, g'_2, \dots$ . We occasionally refer to the standard constructions for the root systems  $D_l$  and  $A_l$ ; see e. g. [Bou], Tables, or [CoSl], Chapter 4, where also the glue vectors are shown. The additional generators of  $K$  will be linear combinations (in practice: sums) of the  $g_i$  and  $g'_j$ . More formally,  $K$  will be the inverse image in  $\langle R \rangle^\#$  of a subgroup  $\mathcal{K} \subset T(R)$  which has to be specified. By Theorem 2.2 of [SchBl], the second projection of  $\mathcal{K}$  is contained in the subgroup  $T(Y) \subset T({}^3Y)$  (this inclusion comes from the equality  $\langle {}^3Y \rangle^\# = \frac{1}{3}\langle Y \rangle^\#$ ). Concretely, this means that the  $g'_j$  to be used are just copies of the  $g_j$ , without factors  $\frac{1}{3}$ . In many cases, a bit more can be said in advance, namely the following.

**Lemma 7** *Assume that  $\det Y = |T(Y)|$  is not divisible by 3. Then  $\mathcal{K}$  is of the form  $\mathcal{K} = \{(x, \varphi x) \mid x \in T(Y)\} \subset T(Y) \times T({}^3Y)$ , for an appropriate isomorphism  $\varphi : T(Y) \rightarrow T(Y)$ .*

The proof is easy and well known from the unimodular case; observe that  $K$  is locally unimodular at all primes  $\neq 3$ .

The following two general arguments will be used without further mentioning in the treatment of the individual cases. In order to ensure that the root system does not enlarge,  $R(K) = R$ , it is sufficient that no new vectors of norm 2 are introduced. Thus the precise condition on  $\mathcal{K}$  is as follows: for any  $x \in \mathcal{K} \setminus \{0\}$ , the *length*, i.e. the minimal norm of a representative vector in  $\langle R \rangle^\#$ , is integral, even, and at least 4. The second general argument deals with the orthogonal group  $O(K)$ . It is intermediate between the Weyl group  $W(R) = W(Y) \times W({}^3Y)$  and the group  $\text{Aut}(Y) \times \text{Aut}({}^3Y)$ . The relevant factor group  $A(K) = O(K)/W(R)$  can be considered as a subgroup of the outer automorphism group (group of diagram automorphisms)  $A(Y) \times A({}^3Y) = A(Y) \times A(Y)$ , acting on the discriminant group

$T(\mathbf{Y}) \times T(\mathbf{Y})$ . Observe furthermore that  $-\text{id}_{\mathbf{X}}$  is not in the Weyl group for  $\mathbf{X} = \mathbf{A}_l$ . In the following,  $S_n$  denotes the symmetric group on  $n$  letters,  $Z_m$  the cyclic group of order  $m$ , and  $D_{i_{2k}}$  the dihedral group of order  $2k$ .

$$\boxed{\mathbf{R} = 6\mathbf{A}_1 6^3\mathbf{A}_1}$$

The discriminant group  $T(\mathbf{Y}) = T(6\mathbf{A}_1)$  is isomorphic to  $Z_2^6$ , with generators  $\bar{g}_i$ ,  $i = 1, \dots, 6$  of norm  $\frac{1}{2}$  (since  $g_i = e_i/2$  in standard coordinates). According to Lemma 7, the code is given by an isomorphism  $\varphi : Z_2^6 \rightarrow Z_2^6$ ; this isomorphism is specified by the images  $\varphi(\bar{g}_i)$ ,  $i = 1, \dots, 6$ . If  $c$  denotes the number of non-zero components of  $\varphi(\bar{g}_i)$ , then the length of  $(g_i, \varphi(\bar{g}_i))$ , equals  $(1 + 3c)/2$ . This number must be even and  $\neq 2$  and thus  $c = 5$ . Therefore  $\varphi(\bar{g}_i)$  is the sum of the  $\bar{g}'_j$ ,  $j \neq \pi(i)$  for some permutation  $\pi \in S_6$ . Since  $S_6$  acts by automorphisms of  ${}^3\mathbf{Y} \subset \mathbf{R}$ , the lattice  $K$  given by  $\varphi$  is unique up to isometry. Every element of  $S_6$ , acting by outer automorphism on  $\mathbf{Y}$ , uniquely extends via  $\varphi$  to  ${}^3\mathbf{Y}$  and thus to the whole lattice. I.e.  $\text{O}(K) \cong \text{W}(\mathbf{R}) \rtimes S_6 \cong Z_2^{12} \rtimes S_6$ , of order  $2^{12} \cdot 6! = 2^{16} \cdot 3 \cdot 5$ .

$$\boxed{\mathbf{R} = 3\mathbf{A}_2 3^3\mathbf{A}_2}$$

We have  $T(\mathbf{A}_2) \cong Z_3$ , with non-zero elements  $\pm\bar{g}_i$ , of length  $\ell(\bar{g}_i) = \frac{2}{3}$ , in the  $i$ -th component ( $i = 1, 2, 3$ ) of  $\mathbf{Y}$ . The glue code  $\mathcal{K}$  is three-dimensional over the field  $\mathbb{F}_3$ . Let  $\mathcal{K}_1$  be the projection onto the first factor  $T(\mathbf{Y}) = Z_3^3$ . First observe that  $\mathcal{K}_1 \neq 0$ ; otherwise, there would be additional roots (of norm 6 in the first factor). Since the length function takes integral values on the second factor, the same must be true for  $\mathcal{K}_1$ . Consequently,  $\mathcal{K}_1$  must be one-dimensional over  $\mathbb{F}_3$ , generated by  $\bar{g}_1 \pm \bar{g}_2 \pm \bar{g}_3$  (of length 2). The sign can be normalized to  $\mathcal{K}_1 = \langle \bar{g}_1 + \bar{g}_2 + \bar{g}_3 \rangle$  by automorphisms of the root system  $\mathbf{Y}$ . The intersection  $\mathcal{K}_2$  of  $\mathcal{K}$  with the second factor  $T({}^3\mathbf{Y}) \cong Z_3^3$  is of dimension 2 and contains no single  $\bar{g}'_2$  (in view of the length function). Clearly, such a subgroup  $\mathcal{K}_2 \subset Z_3^3$  can be normalized to  $\mathcal{K}_2 = \{x_1\bar{g}'_1 + x_2\bar{g}'_2 + x_3\bar{g}'_3 \mid x_j \in \mathbb{F}_3, \sum x_j = 0\} = \langle \bar{g}'_1 - \bar{g}'_2, \bar{g}'_2 - \bar{g}'_3 \rangle$ . The whole glue code  $\mathcal{K}$  is now uniquely determined: the third generator is of the form  $\bar{g}_1 + \bar{g}_2 + \bar{g}_3 + \bar{g}'$ , with  $\bar{g}' \neq 0$  and therefore unique (mod  $\mathcal{K}_2$ ); take e.g.  $\bar{g}' = \bar{g}'_1$ . Since all sign changes except  $-\text{id}$  on all of  $T(\mathbf{R})$  have been used to normalize  $\mathcal{K}$ , the outer automorphism group is  $\langle -\text{id} \rangle \times S_3 \times S_3$ , where the symmetric group  $S_3$  acts by permutation on the first and second factor  $Z_3^3$ . Thus,  $\text{O}(L)$  is of order  $2 \cdot 6^2 \cdot 6^6$ .

$$\boxed{\mathbf{R} = 2\mathbf{A}_3 2^3\mathbf{A}_3}$$

We have  $T(\mathbf{Y}) \cong Z_4^2$ , with generators  $\bar{g}_1, \bar{g}_2$  of length  $\ell(\bar{g}_i) = \frac{3}{4}$ , and  $\ell(2\bar{g}_i) = 1$ . According to Lemma 7,  $\mathcal{K}$  is generated by  $\bar{g}_1 + \varphi(\bar{g}_1)$ ,  $\bar{g}_2 + \varphi(\bar{g}_2)$ , for appropriate elements  $\varphi(\bar{g}_i)$  in the second factor. Since  $\frac{3}{4} + \ell(\varphi(\bar{g}_i))$  must be integral and even, the only choice, up to permuting the two components of the second  $Z_4^2$ , and changing signs, is

$$\varphi(\bar{g}_1) = \bar{g}'_1 + 2\bar{g}'_2, \quad \varphi(\bar{g}_2) = \bar{g}'_2 + 2\bar{g}'_1$$

The glue code thus defined has indeed the property that all its elements have even length at most 4. Every element of the outer automorphism group  $A(Y) \cong Di_8$  of the first factor extends uniquely to the second factor via  $\varphi$ . Thus  $O(K)$  is of order  $2^3 \cdot (4!)^4$ .

$$\boxed{R = D_4 {}^3D_4 2G_2}$$

We have to construct an 8-dimensional lattice  $M$  with  $R(M) = D_4 {}^3D_4$ ; the desired 12-dimensional lattice will then be  $K = M \perp A_2 \perp A_2$ . The discriminant group  $T(D_4)$  has 3 non-zero elements given by glue vectors  $g_1, g_2, g_3 \equiv g_1 + g_2$ , all of norm 1, and permuted transitively by  $A(D_4) \cong S_3$ . As glue vectors of  $K$ , we may take  $g_1 + g'_1, g_2 + g'_2$ , (and thus also  $g_3 + g'_3$ ), which are of norm 4 and satisfy all other requirements. Moreover, this choice is unique up to isomorphism, in view of the transitivity properties of  $A(D_4) \times A(D_4)$ . For the same reason, each element of  $A(D_4) \cong S_3$  acting on the first factor can be extended to an automorphism of  $K$ , i.e.  $A(M) \cong S_3$ . Thus  $|O(K)| = 3! \cdot (2^3 \cdot 4!)^2 \cdot 2 \cdot 12^2$ .

$$\boxed{R = 6G_2}$$

The lattice is the root lattice  $6A_2$  itself, with zero glue code, and orthogonal group  $W(G_2)^6 \rtimes S_6$ , i.e. the wreath product of  $S_6$  and  $Di_{12}$ , of order  $6! \cdot 12^6$ .

$$\boxed{R = A_5 {}^3A_5 G_2}$$

We have to construct a 10-dimensional lattice  $M$  with  $R(M) = A_5 {}^3A_5$ . The cyclic discriminant group  $T(A_5)$  decomposes as  $T(A_5) = \langle \bar{g}_1 \rangle \times \langle \bar{g}_2 \rangle$  where  $g_1$  is of norm  $\frac{3}{2}$  and order 2 (mod  $A_5$ ) and  $g_2$  of norm  $\frac{4}{3}$  and order 3 (mod  $A_5$ ). Because of the desired integrality of  $M$ , the unique choice of (generating) glue vectors is  $g_1 + g'_1, g'_2$ . The automorphism group of  $M$  does not contain  $(\text{id}, -\text{id})$  and thus equals  $\langle -\text{id}_M \rangle \times W(A_5) \times W(A_5)$ . The order of  $O(K) = O(M) \times O(A_2)$  is  $2 \cdot (5!)^2 \cdot 12$ .

$$\boxed{R = A_6 {}^3A_6}$$

$T(A_6) \cong Z_7$  has non-zero elements  $\pm \bar{g}, \pm 2\bar{g}, \pm 3\bar{g}$  of length  $\frac{6}{7}, \frac{10}{7}, \frac{12}{7}$ . The glue code  $\mathcal{K}$  must be generated by  $\bar{g} \pm j\bar{g}'$  for some  $j \in \{1, 2, 3\}$ . Since  $l(j\bar{g}') = \frac{18}{7}, \frac{30}{7}, \frac{36}{7}$ , only the case  $j = 3$  is possible. The two choices of the sign can be transformed into each other by the outer automorphism  $-\text{id}$ , acting on the second factor  $T({}^3A_6)$ . Thus,  $\mathcal{K}$  is unique up to isomorphism, with orthogonal group  $\langle -\text{id} \rangle \times W(A_6) \times W(A_6)$ , of order  $2 \cdot (7!)^2$ .

$$\boxed{R = D_6 {}^3D_6}$$

$T(D_6) \cong Z_2^2$  is generated by elements  $\bar{g}_1, \bar{g}_2$  of length  $\ell(\bar{g}_i) = \frac{3}{2}$ , with  $\ell(\bar{g}_1 + \bar{g}_2) = 1$ . Again, the structure of  $\mathcal{K}$  is described by Lemma 7. The only solution, up to permuting  $\bar{g}'_1$  and  $\bar{g}'_2$  by the outer automorphism of  ${}^3D_6$ , is

$$\mathcal{K} = \langle \bar{g}_1 + \bar{g}'_1, \bar{g}_2 + \bar{g}'_2 \rangle.$$

The automorphism group is  $Z_2 \cdot W(D_6) \times W(D_6)$ , of order  $2 \cdot (2^5 \cdot 6!)^2$ .

$$\boxed{R = E_6 \ ^3E_6}$$

Since  $T(E_6)$  is of order 3, the glue code is cyclic, generated by  $\bar{g} + \bar{g}'$  say. Since every  $\bar{g}'$  in the second factor has integral length, also  $\bar{g}$  must be of integral length. This forces  $\bar{g} = 0$ , and consequently  $\bar{g}' \neq 0$ , i.e.  $\langle \bar{g}' \rangle = T(^3E_6)$ . Thus  $K = E_6 \perp E_6^\sharp$ , and  $O(K) = O(E_6) \times O(E_6)$ ,  $A(K) \cong Z_2^2$ ,  $|O(K)| = 2^2 \cdot (2^7 \cdot 3^4 \cdot 5)^2$ .

We summarize our results in the following table, where we list all our lattices  $K$  together with the Coxeter number  $h$ , the structure of the outer automorphism group, and the order of  $O(K)$ . Using the orders of the orthogonal groups, one can readily confirm our enumeration by the mass formula. The mass of  $\mathcal{H}$  is equal to  $11 \cdot 13^2 \cdot 41 \cdot 61 / 2^{19} \cdot 3^8 \cdot 5^2 \cdot 7^2$ .

lattice	$h$	root system	outer group	order of $O(K)$
$K_1 = C$	–	$\emptyset$		$2^{10} 3^7 5^7$
$K_2$	2	$6A_1 \ 6^3A_1$	$S_6$	$2^{16} 3^{25}$
$K_3$	3	$3A_2 \ 3^3A_2$	$Z_2 \times S_3 \times S_3$	$2^9 3^8$
$K_4$	4	$2A_3 \ 2^3A_3$	$Di_8$	$2^{15} 3^4$
$K_5$	6	$D_4 \ ^3D_4 \ 2G_2$	$S_3 \times Z_2$	$2^{18} 3^5$
$K_6$	6	$6G_2$	$S_6$	$2^{12} 3^8 5$
$K_7$	6	$A_5 \ ^3A_5 \ G_2$	$Z_2$	$2^{11} 3^5 5^2$
$K_8$	7	$A_6 \ ^3A_6$	$Z_2$	$2^9 3^4 5^2 7^2$
$K_9$	10	$D_6 \ ^3D_6$	$Z_2$	$2^{19} 3^4 5^2$
$K_{10}$	12	$E_6 \ ^3E_6$	$Z_2 \times Z_2$	$2^{16} 3^8 5^2$

Table 1: The genus  $\mathcal{H}_{12}(3^6)$  of lattices of level 3, dimension 12, determinant  $3^6$

## 4 Lattices of level 2: summary of results

The dimension of a lattice of level 2 is known to be divisible by 4, its determinant is of the form  $2^{2\delta}$  with  $\delta \leq n/2$ , and  $n$  is divisible by 8 if  $\delta = 0$  or  $\delta = n/2$ . Conversely, for any such pair  $(n, \delta)$  there exists exactly one genus of totally even 2-elementary lattices with these parameters.

This section complements the results of [SchVe94] by finishing the classification lattices of level 2 in dimensions  $n$  at most 16. In view of the unique

decomposition of an arbitrary (positive definite) lattice as an orthogonal sum of indecomposable ones, which in our case have to be of level  $\leq 2$  as well, the classification problem immediately reduces to the case  $n = 16$ . It is shown that, except for the Barnes-Wall lattice ( $n = 16, \delta = 4$ ), all these lattices are reflective, i.e. their root system is of full rank (cf. [SchB196]). The root system  $R(L)$  consists of short roots of norm 2 and long roots of norm 4. The investigation of the possible root systems, in particular the relation between short and long roots, is more complicated than in the case  $\delta = 4$  treated in [SchV] or the case  $\ell = 3, n = 12, \det = 3^6$  treated above. It is carried out using ordinary theta series and theta series with harmonic coefficients (of degree 2). One uses the well known transformation formula relating the theta series of  $L$  and  $L^\#$ , which can be rewritten in terms of the so called Fricke involution  $z \mapsto -\frac{1}{2z}$  acting on modular forms, and the explicit knowledge of modular forms for  $\Gamma_0(2)$  of weight  $\leq 10$ . It turns out that a certain modified Coxeter number is the same for all components of  $R(L)$ , which gives serious restrictions leading to an explicit list of possible  $R(L)$ 's.

Recall the root system  $R(L)$  of a totally even 2-elementary lattice can be described as  $R(L) = R_{sh}(L) \cup R_{lo}(L)$  (“short” and “long” roots) with  $R_{sh}(L) = \{v \in L \mid (v, v) = 2\}$  and  $R_{lo}(L) = \{2w \mid w \in L^\#, (w, w) = 1\}$ . The possible irreducible components of  $R(L)$  are  ${}^\alpha A_\ell, {}^\alpha D_\ell$  ( $\ell \geq 4$ ),  ${}^\alpha E_\ell$  ( $\ell = 6, 7, 8$ ),  $C_\ell, {}^2B_\ell, F_4$ , where the upper index  $\alpha$  denotes scaling and in our present case equals 1 or 2.

**Theorem 8** *All 16-dimensional totally even 2-elementary lattices of determinant  $\neq 2^8$  are reflective. Possible root system of such lattices of determinant  $2^\delta, \delta = 3, 2, 1$  have the following properties:*

a) *The numbers of short roots  $|R_{sh}|$  and of long roots  $|R_{lo}|$  are related by*

$$|R_{sh}| = \begin{cases} 2|R_{lo}| + 32 & \text{if } \delta = 3 \\ 4|R_{lo}| + 96 & \text{if } \delta = 2 \\ 8|R_{lo}| + 224 & \text{if } \delta = 1 \end{cases}$$

b) *There is a natural number  $h_\delta$  (“modified Coxeter number of  $R$ ”) with the following properties, where we set  $t = 2^{4-\delta}$ :*

- (1) *all irreducible components contained in  $R_{sh}$  have the Coxeter number  $h_\delta$*
- (2) *all irreducible components contained in  $R_{lo}$  have the Coxeter number  $h_\delta/t$*
- (3) *all components of type  ${}^2B_\ell$  have the same  $\ell$  and  $h_\delta = 2 + 2(\ell - 1)t$*
- (4) *all components of type  $C_m$  have the same  $m$  and  $h_\delta = 2(m - 1) + 2t$*

(5) components of type  $F_4$  could be in  $R$  only if  $h_\delta = 6(t + 1)$

Elementary, but lengthy considerations lead to the following

**Proposition 9** *There are exactly  $17 + 11 + 6$  root systems of rank 16 satisfying the conditions of Theorem 8. These are the root systems shown in Tables 2, 3 and 4 below.*

Proceeding similarly as in the previous section one can show with some patience the following theorem which, together with the previous Proposition 9 completes the proof of theorem 2.

**Theorem 10** *Each root system  $R$  of Theorem 8 and Proposition 9 is realizable as  $R(L)$ , for an appropriate lattice  $L$ . The lattice  $L$  is uniquely determined (up to isometry) by its root system.*

The following three tables summarize the main properties of the lattices of Theorem 10 resp. Theorem 2: The “Coxeter number”  $h_\delta$  according to Theorem 8, the root system, the index  $[L : \langle R(L) \rangle]$ , the order of the orthogonal group  $O(L)$ , and the order of the “outer automorphism group”  $A(L) := O(L)/W(L)$ , where  $W(L)$  denotes the Weyl group (of  $R(L)$ ).

no.	$h_\delta$	root system	$ A(L) $	$ O(L) $
1	2	$16A_1$	$2^{10} \cdot 3^2 \cdot 5 \cdot 7$	$2^{26} \cdot 3^2 \cdot 5 \cdot 7$
2	4	$4A_3 4^2A_1$	$2^6 \cdot 3$	$2^{22} \cdot 3^5$
3	6	$2D_4 4C_2$	$2^4 \cdot 3$	$2^{28} \cdot 3^3$
4	6	$2A_5 2^2A_2 C_2$	$2^3$	$2^{16} \cdot 3^6 \cdot 5^2$
5	8	$2D_5 2^2A_3$	$2^3$	$2^{23} \cdot 3^4 \cdot 5^2$
6	8	$A_7 {}^2A_3 2C_3$	$2^2$	$2^{20} \cdot 3^5 \cdot 5 \cdot 7$
7	10	$D_6 C_4 2^2B_3$	2	$2^{25} \cdot 3^5 \cdot 5$
8	10	$A_9 {}^2A_4 {}^2B_3$	2	$2^{16} \cdot 3^6 \cdot 5^3 \cdot 7$
9	10	$4C_4$	$2^3 \cdot 3$	$2^{31} \cdot 3^5$
10	12	$E_6 {}^2A_5 C_5$	2	$2^{20} \cdot 3^7 \cdot 5^3$
11	14	$2C_6 {}^2B_4$	2	$2^{28} \cdot 3^5 \cdot 5^2$
12	14	$D_8 2^2B_4$	2	$2^{29} \cdot 3^4 \cdot 5 \cdot 7$
13	16	$D_9 {}^2A_7$	1	$2^{23} \cdot 3^6 \cdot 5^2 \cdot 7^2$
14	18	$E_7 {}^2B_5 F_4$	1	$2^{25} \cdot 3^7 \cdot 5^2 \cdot 7$
15	18	$C_8 2F_4$	2	$2^{30} \cdot 3^6 \cdot 5 \cdot 7$
16	22	$C_{10} {}^2B_6$	1	$2^{28} \cdot 3^6 \cdot 5^3 \cdot 7$
17	30	$E_8 {}^2B_8$	1	$2^{29} \cdot 3^7 \cdot 5^3 \cdot 7^2$

Table 2: The genus  $\mathcal{H}_{16}(2_{II}^6)$  of lattices of level 2, dimension 16, determinant  $2^6$

no.	$h_\delta$	root system	$ A(L) $	$ O(L) $
1	6	$4D_4$	$2^4 \cdot 3^2$	$2^{28} \cdot 3^6$
2	8	$2A_7 2^2A_1$	$2^3$	$2^{19} \cdot 3^4 \cdot 5^2 \cdot 7^2$
3	10	$2D_6 2C_2$	$2^2$	$2^{26} \cdot 3^4 \cdot 5^2$
4	12	$2E_6 2^2A_2$	$2^3$	$2^{19} \cdot 3^{10} \cdot 5^2$
5	12	$A_{11} 2A_2 C_3$	2	$2^{16} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11$
6	14	$D_8 2C_4$	2	$2^{29} \cdot 3^4 \cdot 5 \cdot 7$
7	18	$E_7 C_6 2B_3$	1	$2^{24} \cdot 3^7 \cdot 5^2 \cdot 7$
8	18	$D_{10} 2^2B_3$	2	$2^{26} \cdot 3^6 \cdot 5^2 \cdot 7$
9	22	$2C_8$	2	$2^{31} \cdot 3^4 \cdot 5^2 \cdot 7^2$
10	30	$E_8 2F_4$	2	$2^{29} \cdot 3^9 \cdot 5^2 \cdot 7$
11	30	$C_{12} F_4$	1	$2^{29} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11$

Table 3: The genus  $II_{16}(2_{II}^4)$  of lattices of level 2, dimension 16, determinant  $2^4$

no.	$h_\delta$	root system	$ A(L) $	$ O(L) $
1	14	$2D_8$	2	$2^{29} \cdot 3^4 \cdot 5^2 \cdot 7^2$
2	16	$A_{15} 2A_1$	2	$2^{17} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$
3	18	$2E_7 C_2$	2	$2^{24} \cdot 3^8 \cdot 5^2 \cdot 7^2$
4	22	$D_{12} C_4$	1	$2^{28} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11$
5	30	$C_8 E_8$	1	$2^{29} \cdot 3^7 \cdot 5^3 \cdot 7^2$
6	46	$C_{16}$	1	$2^{31} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$

Table 4: The genus  $II_{16}(2_{II}^2)$  of lattices of level 2, dimension 16, determinant  $2^2$

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